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ON GROUPS OF HYPERBOLIC LENGTH

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ABSTRACT

Upper and lower bounds are established for the maximum length of a chain of subgroups in a finite classical linear group. Also, it is proved that, for each prime p, all but finitely many finite Lie type groups in characteristic p have a longest chain which passes through a maximal parabolic.

1. Introduction

There are a number of ways for one to measure the "size" of a finite group G (the order of G being the most obvious). Peter Cameron has suggested that for many purposes, the most indicative measure is the **length** of G, which is defined as follows: *If G is a finite group, $\ell(G)$ is the length of a longest strictly descending chain of subgroups in G.* This work is concerned with chains of subgroups in classical groups. We obtain the following bounds on the lengths of classical linear groups:

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THEOREM 5.1: *Let $G = \mathcal{I}(n,q)$ be a classical matrix group with minimal field of definition $\mathbb{F}_q$ where $q = p^m$, $q > 11$. Suppose further that G is a split BN-pair of rank $r(G)$ with $B = UH = N_G(U)$, $U \in \mathrm{Syl}_p(G)$ and H a Cartan subgroup of G. Then one of the following holds:*

(i) $r(G) + \log_p(|U|) + \Omega(|H|) \leq \ell(G) \leq r(G) + \log_p(|U|) + \log_2(|H|)$

or

(ii) $E(G) \in \{\mathrm{SU}_2(p), \Omega_3(p), \mathrm{Sp}_4(p), \Omega_4^+(p), \Omega_5(p)\}$ *for p a Mersenne prime.*

Theorem 5.1 is obtained as a corollary of the following result which provides a minimal list of "unavoidable" longest subgroups of classical linear groups. Here a maximal subgroup M is called a longest subgroup of G if $\ell(G) = \ell(M) + 1$.

THEOREM 1.1: *Let $G = G(p^m)$ be a classical linear group with natural module V of dimension n over $\mathbb{F} = \mathbb{F}_q$, $q = p^{\overline{m}}$, $\overline{m} \in \{m, 2m\}$. Suppose that $G_0 \lhd G$ with G_0 quasisimple and assume that G does not induce a graph automorphism of G_0 if G_0 is of type SL_n. Then there exists a maximal subgroup M of G with $\ell(M) + 1 = \ell(G)$ satisfying one of the following:*

(1) $M = N_G(M_0)$, *where M_0 is a maximal parabolic subgroup of G_0; or*

(2) *G stabilizes a non-degenerate form on V, $V = V_1 \perp V_2$ and $M = \mathrm{Stab}_G(\{V_1, V_2\})$. Furthermore, if $\dim(V_i) = n_i$ and G is orthogonal, then n_2 is even and, if n is odd, then $n_1 = 1$; or*

(3) *$\mathbb{K}$ is a field extension of $\mathbb{F}$ with $(\mathbb{K}\colon \mathbb{F}) = r$, where r is the smallest prime divisor of n, and M is the stabilizer in G of a $\mathbb{K}$-linear structure on V. Moreover, either*

   (i) *G is of type SL_{rm} (r prime) or SU_{rm} (r prime, rm odd) and $F^*(M)$ is of type SL_m or SU_m respectively with $m \geq 1$, or*

   (ii) *G is of type Sp_{4s} or O_{4s}^- and $F^*(M)$ is of type Sp_{2s} or O_{2s}^-, or*

   (iii) *G is of type Sp_{2s} or O_{2s}^-, (s odd) and $F^*(M)$ is of type SU_s;*

(4) *G is of type O^+, $[G\colon M] = 2$ and $G = \langle M, \gamma \rangle$ where γ induces a graph automorphism of order 2 on $M^{(\infty)}$. Moreover, $M_0 \leq M$ with M_0 the normalizer in M of a maximal totally isotropic subspace of V and $\ell(M) = \ell(M_0) + 1$; or*

(5) *G is of type $\mathrm{SL}_2(p)$, $p \in \{5, 7, 11, 19, 29\}$.*

Our other application of Theorem 1.1 requires some definitions.

Definition 1.2: Let G be any finite group of Lie type with $\Omega = E(G)$ quasisimple. The **parabolic length** of G, denoted by $\ell_\pi(G)$, is defined to be $\max\{\ell(P) + 1\}$, where P ranges over all parabolic subgroups of G. If $\ell(\Omega) = \ell_\pi(\Omega)$ we will say that G has parabolic length for every $\Omega \leq G \leq \Gamma$ (see below). Otherwise, we say that G has **hyperbolic length** (Our definition for groups of parabolic length differs slightly from that in [2].)

Remark: According to Definition 1.2, G has parabolic length if and only if G contains a longest subgroup M of type (1) or (4) in Theorem 1.1.

In [25] it is shown that for any fixed prime p, there exists a number $F(p)$ such that any quasisimple Lie type group in characteristic p whose minimal field of definition has cardinality at least $p^{F(p)}$ must have parabolic length. Alternatively, quasisimple groups of hyperbolic length in characteristic p only occur over fields of cardinality smaller that $p^{F(p)}$. By itself, this does not imply that there are only **finitely** many quasisimple groups of hyperbolic length. However, Theorem 1.1 permits us to establish a bound on the length of groups of hyperbolic length, which in turn easily yields

THEOREM 4.4: *For each prime p, there are only finitely many (possibly 0) finite Lie type groups G in characteristic p with G of hyperbolic length.*

2. Preliminaries

For the most part, our notation will be consistent with the notation found in [15]. Indeed, the results therein are critical to our analysis.

Let (V, f) be a n-dimensional vector space over the field of q elements together with an associated form f (possibly trivial). We shall refer to G as a **classical matrix group** if G is a subgroup of the full isometry group of (V, f), and will sometimes write $G = \mathcal{I}(n, q)$.

We let Ω denote any of the groups

$$\mathrm{SL}(V), \mathrm{SU}(V), \mathrm{Sp}(V), \Omega(V), \Omega^+(V), \Omega^-(V),$$

and Γ any of the groups

$$\Gamma L(V), \Gamma U(V), \Gamma\,\mathrm{Sp}(V), \Gamma O(V), \Gamma O^+(V), \Gamma O^-(V).$$

For our purposes, a group G is said to be **classical linear** if

$$\overline{\Omega} \leq G \leq \overline{\Gamma}$$

where $\overline{X}$ denotes reduction of X modulo a group of scalars.

In addition, we will refer to G as being **of type** L, U, S, O^ϵ with $\epsilon \in \{\circ, +, -\}$ according to whether Ω is respectively $\mathrm{SL}(V)$, $\mathrm{SU}(V)$, $\mathrm{Sp}(V)$, $\Omega^\epsilon(V)$ (here O° denotes an odd dimensional orthogonal group).

For small values of p, the precise list of Lie type groups of hyperbolic length in characteristic p may be obtained and indeed this has been done independently of our main theorem in [24] and [5] for $p \leq 29$. Thus in this paper we may assume the following:

LEMMA 2.1: $p \geq 31$.

Certain other general results of Brozovic play a critical role in the proof. Throughout the ensuing discussion we shall assume the following:

($\star$) $G = G(q)$ is a finite quasisimple group of Lie type defined over $\mathbb{F} = \mathbb{F}_q$, $q = p^m$, $p \geq 31$. $G = M_0 \supset M = M_1 \supset M_2 \supset \cdots \supset M_r = \{e\}$ is a strictly descending chain of subgroups in G with $\ell(G) = r$ (so $\ell(G) = \ell(M) + 1$).

THEOREM 2.2: *We may choose M so that either $F^*(M) \neq F(M)$ or $M = N_G(T)$ for T some non-split maximal torus of G.*

Proof: This follows from Theorem 5.1 of [4]. ∎

THEOREM 2.3: *If $F^*(M) \neq F(M)$ and L is a quasisimple subnormal subgroup of M, then either*

(a) *$L = L(q')$ is a finite quasisimple group of Lie type defined over $\mathbb{F}_{q'}$, $q' = p^{m'}$; or*

(b) *$G/Z(G) \cong L_2(p)$ and $MZ(G)/Z(G) \cong A_5$.*

Moreover in case (b), *$\ell(M) = \ell(N_G(T))$ for T some maximal torus of G.*

Proof: This follows from the main theorems of [2] and [3]. ∎

Note that since T is solvable, $\ell(T) = \Omega(|T|)$ where $\Omega(n)$ is defined as follows:

Definition 2.4: Let $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ be the prime factorization of the positive integer n. Then

$$\Omega(n) = \alpha_1 + \cdots + \alpha_s.$$

The following properties of the function $\Omega(n)$ are evident.

PROPOSITION 2.5:

(a) $\Omega(n \cdot m) = \Omega(n) + \Omega(m)$;

(b) $\Omega(n) \leq \log_2(n)$ *for all* $n \geq 1$.

THEOREM 2.6: *Suppose L_i is a subnormal subgroup of M_i $(0 \le i \le r)$ with L_i a quasisimple group of Lie type in characteristic p. If L_i has parabolic length, so does G.*

Proof: If $i = 0$, then $L = E(G)$ and this follows from the definition of parabolic length. If $i > 0$, this follows by induction from the main theorem of [2]. ∎

Henceforth we assume the following:

($\star\star$) Hypothesis ($\star$) holds and G is a counterexample to Theorem 1.1 such that if M^* is a maximal subgroup of G, then Theorem 1.1 holds in every section of M^*.

Combining the above theorems we obtain the following result:

PROPOSITION 2.7: *We may assume the following:*

(a) $F^*(M) \neq F(M)$.

(b) *If L_i is a quasisimple subnormal subgroup of M_i, $1 \le i \le r$, then L_i is a group of Lie type in characteristic p and L_i has hyperbolic length.*

(c) *If L_i is a quasisimple subnormal subgroup of M_i, $1 \le i \le r$, and L_i is a classical linear group, then L_i has a maximal subgroup K_i with $\ell(L_i) = \ell(K_i) + 1$ and with the pair (L_i, K_i) satisfying conclusion (2) or (3) of Theorem 1.1.*

Proof: If $F^*(M) = F(M)$, then by Theorem 2.3, $M = N_G(T)$ for T a non-split maximal torus of G. Then by Theorem 1 of [1], G has type $\mathrm{SL}_r(q)$ or $\mathrm{SU}_r(q)$ with r prime and T is a Coxeter torus in G. But then conclusion (3) of Theorem 1.1 holds, contrary to assumption. Thus (a) is valid. Part (b) follows by induction from Theorems 2.3 and 2.6. Part (c) is immediate from (b) by our minimal choice of G in ($\star\star$). ∎

The following fact, established in [15], is very useful.

PROPOSITION 2.8: *Suppose $G = \overline{\Omega}$ and M satisfies conclusion* (1), (2) *or* (3) *of Theorem* 1.1. *If $\Omega = \Omega^+_{2n}(q)$, assume that M is not the stabilizer of a maximal totally isotropic subspace of V. Then $M^{\overline{\Gamma}} = M^{\overline{\Omega}}$ (here $M^{\overline{\Gamma}}$ denotes the $\overline{\Gamma}$ class of M). Thus if $\overline{\Omega} \le X \le \overline{\Gamma}$, then $X = \overline{\Omega} N_X(M)$.*

Proof: The conjugacy of members of $\mathcal{C}_i$, $1 \le i \le 3$, is discussed in Sections 4.1–4.3 of [15] and the results are tabulated in Column V of Tables 3.5.A–3.5.F. The second conclusion follows from the first by a Frattini argument. ∎

LEMMA 2.9: *Let G be a classical linear group containing Ω and satisfying the hypotheses of Theorem* 1.1. *Then G satisfies the conclusion of Theorem* 1.1 *if and only if Ω does.*

Proof: Suppose Theorem 1.1 holds for (Ω, M). If $M^G = M^\Omega$, then $G = \Omega N_G(M)$ and $\ell(G) = \ell(N_G(M)) + 1$, and we are done. If not, then by Proposition 2.8, $\Omega = \Omega_{2n}^+(q)$, M is the stabilizer of a maximal totally isotropic subspace of V and G contains γ inducing an involutory graph automorphism on Ω, whence case (4) of Theorem 1.1 holds for G.

Now suppose Theorem 1.1 holds for G. If case (4) holds, then we may replace (G, M) by (M, M_0). Then changing notations if necessary, we have in all cases that $\Omega \not\subseteq M$ and $\ell(G) = \ell(M) + 1$, whence $\ell(\Omega) = \ell(M \cap \Omega) + 1$ and the theorem holds for Ω. ∎

LEMMA 2.10: *Let G be a classical subgroup of $\Gamma L(V)$ containing $\Omega(V)$ and satisfying the hypotheses of Theorem* 1.1. *Let Z be a subgroup of $Z(\mathrm{GL}(V)) \cap G$. Then G satisfies the conclusion of Theorem* 1.1 *if and only if $\overline{G} = G/Z$ does.*

Proof: This is immediate from the fact that $Z \subseteq M$ for any M in the conclusion of Theorem 1.1. ∎

COROLLARY 2.11: *We may (and shall) assume that $G = \Omega$.*

Definition 2.12: We shall call an $\mathbb{F}[M]$-module V induced if $V = \mathrm{ind}_H^M(U)$ for some proper subgroup H of M and $\mathbb{F}[H]$-module U (see [9], p. 228). We shall call an $\mathbb{F}[M]$-module V tensor-induced if $V = \otimes\,\mathrm{ind}_H^M(U)$ for some proper subgroup H of M and $\mathbb{F}[H]$-module U (see [9], p. 333).

For the analysis of the induced and tensor induced cases, the following fact is useful.

Definition 2.13: Let $n = c_r 2^r + \cdots + c_1 2 + c_0$ with $c_i \in \{0, 1\}$, $c_r = 1$. Suppose $\mathcal{C} = \{c_0, c_1, \cdots, c_r\} = \mathcal{C}_1 \cup \mathcal{C}_2$ is a partition of $\mathcal{C}$ and

$$n_i = \sum_{c_j \in \mathcal{C}_i} c_j 2^j, \quad i = 1, 2.$$

We say that $n = n_1 + n_2$ is a **dyadic splitting** of n. We call it a proper dyadic splitting if $n_1 \neq 0 \neq n_2$.

THEOREM 2.14: *Let $G = \mathrm{Sym}(n)$. If $n \neq 2^r$ and $n = n_1 + n_2$ is a proper dyadic splitting of n, then G has a maximal intransitive subgroup $M \cong \mathrm{Sym}(n_1) \times \mathrm{Sym}(n_2)$ with $\ell(G) = \ell(M)+1$. If $n = 2^r \geq 4$ and $m = n/2$, then G has a maximal imprimitive subgroup $M \cong \mathrm{Sym}(m) \wr \mathrm{Sym}(2)$ with $\ell(G) = \ell(M) + 1$.*

Proof: This is essentially the main theorem of [7]. ∎

The organization of the proof of Theorem 1.1 in Section 3 is as follows. Assuming that (G, M) satisfies $(\star\star)$ with $G = \Omega$, we consider the possibilities for M, following Aschbacher's organization of cases, modified by Seitz to include stabilizers of twisted tensor decompositions.

Definition 2.15: E acts **absolutely tensor indecomposably** on V (resp. **absolutely tensor decomposably**) on V if E acts tensor indecomposably (resp. tensor decomposably) on $V^{\mathcal{L}} = V \otimes_{\mathbb{F}} \mathcal{L}$ for every (resp. some) finite extension $\mathcal{L}$ of $\mathbb{F}$. E acts **absolutely irreducibly** (resp. **absolutely reducibly**) on V if E acts irreducibly (resp. reducibly) on $V^{\mathcal{L}}$ for every (resp. some) finite extension $\mathcal{L}$ of $\mathbb{F}$.

In Lemmas 3.1–3.5, we assume $E = E(M)$ acts absolutely reducibly on V and we show that for some M^* with $\ell(M) = \ell(M^*)$, conclusion (2) or (3) of Theorem 1.1 holds, contrary to assumption. In the next three lemmas, we assume $E = E(M)$ acts absolutely tensor decomposably on V and find M^* with $\ell(M) = \ell(M^*)$ and $E(M^*)$ acting absolutely reducibly on V. We are then reduced to the case where $E = E(M)$ is a quasisimple group of Lie type acting absolutely irreducibly and absolutely tensor indecomposably on V. If E is classical, we argue that M is the centralizer of a field, graph, or graph-field automorphism on Ω and then reach an easy contradiction in this case. Finally we argue that E cannot be exceptional, yielding a final contradiction.

Definition 2.16: If $V = V_1 \perp V_2$ satisfies the conditions of conclusion (2) of Theorem 1.1, we shall call it an **admissible (orthogonal) splitting** of V. If the extension $\mathbb{K}/\mathbb{F}$ with $(\mathbb{K}\colon \mathbb{F}) = r$ and the associated subgroup M of $\Gamma L_{\frac{n}{r}}(q^r)$ satisfies Conclusion (3) of Theorem 1.1, we shall speak of an **admissible field extension**.

If M stabilizes a $\mathbb{K}$-linear structure of V there is an associated embedding of $\mathbb{K}$ into Γ. We denote this embedding by $\rho(\mathbb{K})$, so that $M = C_G(\rho(\mathbb{K}))$.

3. The proof of Theorem 1.1

Throughout this section we assume that G and the chain

$$G = M_0 \supset M = M_1 \supset M_2 \supset \cdots \supset M_r = \{e\}$$

satisfy Hypothesis $(\star\star)$. We consider the possible structures of M. In the first two lemmas we treat the case that $E(M)$ acts reducibly on V. We fix $E = E(M)$.

LEMMA 3.1: *Suppose V is induced as an $\mathbb{F}[M]$-module. Then either $V = V_1 \perp V_2$ with (V_1, f_1) isometric to (V_2, f_2) (here $f_i = f|_{V_i}$ and f is the form associated to G), or there exists a maximal subgroup M^* with $\ell(M^*) = \ell(M)$ and $V = W_1 \perp W_2$ as an $\mathbb{F}[E(M^*)]$-module.*

Proof: Suppose not. According to Table 4.2.A from [15] we have $V = V_1 \perp \cdots \perp V_s$ as $\mathbb{F}[E]$-module with $s \geq 3$. For an arbitrary orthogonal decomposition $U_1 \perp \cdots \perp U_k$ of V, we denote by $N_G(\{U_1, \ldots, U_k\})$ the subgroup of G which permutes the set $\{U_1, \ldots, U_k\}$. Similarly, $C_G(\{U_1, \ldots, U_k\})$ shall denote the subgroup of G that fixes each of the U_i. By Corollary 4.2.2 of [15] we have

$$N_G(\{V_1, \ldots, V_s\})/C_G(\{V_1, \ldots, V_s\}) \cong \mathrm{Sym}(s).$$

If $s = s_1 + s_2$ is a proper dyadic splitting, set

$$W_1 = V_1 \perp \cdots \perp V_{s_1}, W_2 = V_{s_1+1} \perp \cdots \perp V_s. \tag{1}$$

Set $Y = N_M(\{W_1, W_2\})$ and observe that

$$Y/C_G(\{V_1, \ldots, V_s\}) \cong \mathrm{Sym}(s_1) \times \mathrm{Sym}(s_2).$$

By Theorem 2.14, $\ell(\mathrm{Sym}(s)) = \ell(\mathrm{Sym}(s_1)) + \ell(\mathrm{Sym}(s_2)) + 1$ and it follows that $\ell(M) = \ell(Y) + 1$. Set $M^* = N_G(\{W_1, W_2\})$. As $\mathrm{Stab}_M(W_i)$ acts imprimitively on W_i we have $M^* \neq M \cap M^* = Y$ and so $\ell(M^*) \geq \ell(M)$ and the result holds.

Otherwise we may suppose $s = 2^r$, $r \geq 2$. Set

$$W_1 = V_1 \perp \cdots \perp V_{\frac{s}{2}}, W_2 = V_{\frac{s}{2}+1} \perp \cdots \perp V_s. \tag{2}$$

Then W_1 and W_2 are isometric. By Theorem 2.14,

$$\ell(\mathrm{Sym}(s)) = \ell\left(\mathrm{Sym}\left(\frac{s}{2}\right) \wr \mathbb{Z}_2\right) + 1,$$

and it again follows that

$$\ell(M) = \ell\left(N_M\left(\{W_1, W_2\}\right)\right) + 1.$$

Letting $M^* = N_G(\{W_1, W_2\})$, we are done as before. ∎

LEMMA 3.2: *E acts irreducibly on V.*

Proof: Suppose not. By Proposition 2.7, $E \neq \{1\}$.

Suppose E stabilizes a proper isotropic subspace V_1 of V. If G is not of type L, Theorem 1 of [1] implies G is of type U, S or O^+ and $V = V_1 \oplus V_2$ with $V_1 \cong V_2$ totally isotropic subspaces of V. Furthermore, M must contain a subgroup D of index 2 with $D \subset P$ for some maximal parabolic subgroup P of G and $L \cap F(P) = \{1\}$. Then clearly $\ell(M) < \ell(P)$, a contradiction. If G is of type L, then $V = V_1 \oplus \cdots \oplus V_k$ as an $\mathbb{F}[E]$-module. By Lemma 3.1 we may replace M by M^* (if necessary) so that $V = V_1 \perp V_2$ as an $\mathbb{F}[E]$-module. As above, there is a subgroup of index at most 2 and contained in a maximal parabolic subgroup of G, a contradiction.

Now suppose E stabilizes a proper non-degenerate subspace of V. By Theorem 1 of [1] together with an application of Lemma 3.1 (and replacement of M by M^*, if necessary), we may assume $V = V_1 \perp V_2$ as an $\mathbb{F}[E]$-module. Set $\dim_{\mathbb{F}}(V_i) = n_i$. As Conclusion (2) of Theorem 1.1 does not hold, we have $G = \Omega^\epsilon(V)$ and we may assume that either $n_1 = n_2$ are odd or n and n_1 are odd but $n_1 \neq 1$. Then $n \geq 5$ and we may always assume that n_1 is odd and $n_1 \geq 3$. Set $N_i = \mathrm{Stab}_M(V_i)$. By induction, there exists, for n_i odd and $n_i > 1$, $L_i = \mathrm{Stab}_{N_i}(\{V_{i1}, V_{i2}\})$ where $V_i = V_{i1} \perp V_{i2}$, $\dim_{\mathbb{F}}(V_{i1}) = 1$ and $\ell(N_i) = \ell(L_i) + 1$. If n is odd set

$$W_1 = V_{11}, \quad W_2 = V_{12} \perp V_2$$

and $M^* = \mathrm{Stab}_G(\{W_1, W_2\})$. Then $\ell(M^*) \geq \ell(M)$ and M^* satisfies Theorem 1.1(2), a contradiction. If n is even, set

$$W_1 = V_{11} \perp V_{21}, \quad W_2 = V_{12} \perp V_{22}$$

and $M^* = \mathrm{Stab}_G(\{W_1, W_2\})$ (here $V_{22} = \{0\}$, if $\dim_{\mathbb{F}}(V_2) = 1$). Again we have a contradiction. ∎

Definition 3.3: Let $G = G(q)$ be a classical linear group of hyperbolic length with natural module V.

(1) We say G is **split hyperbolic** if G has a longest subgroup M such that $E(M)$ acts reducibly on V.

(1) We say G that G is **non-split hyperbolic** if

 (i) G is not split hyperbolic and there is a longest subgroup M of G such that $E(M)$ acts irreducibly but not absolutely irreducibly on V; or

 (ii) G is of type $\mathrm{GL}_1^\epsilon(p^m)$ or $O_2^\epsilon(p^m)$; or

 (iii) $\Omega_4^+(q) \leq G \leq O_4^+(p^m)$ (or the analogous projective groups) and neither subnormal $(S)L_2(p^m)$ subgroup has parabolic length; or

 (iv) $G' = [G,G] \cong (S)L_2(p)$, $p \in \{5,7,11,19,29\}$.

Henceforth we may assume that if M^* is any longest subgroup of G, then $E(M^*)$ acts irreducibly on V.

The following result will be invoked in Section 4. For this reason we formulate it without the assumption that G is a counterexample to Theorem 1.1.

PROPOSITION 3.4: *Suppose that G is a classical subgroup of $\mathrm{GL}(V)$ with $E(G)$ quasisimple and V a vector space over the field $\mathbb{F}$. Suppose that every proper classical subgroup of G satisfies Theorem 1.1. Furthermore, suppose that G is non-split hyperbolic and that M is a longest subgroup of G with M the stabilizer in G of a $\mathbb{K}$-linear structure on V for some proper field extension $\mathbb{K}$ of $\mathbb{F}$. Let*

$$G = M_0 > M = M_1 > \cdots > M_s > \cdots > M_t = \{e\}$$

be a chain with $\ell(G) = t$. Then there exists a proper tower of fields

(†) $\mathbb{F} = \mathbb{K}_0 \subset \mathbb{K}_1 \subset \cdots \subset \mathbb{K}_s$

such that M_i is the stabilizer in G of a $\mathbb{K}_i$-linear structure on V for $1 \leq i \leq s$ and one of the following holds:

(1)

 (a) *$E(G) = \mathrm{SL}(V)$, $\mathrm{SU}(V)$ ($\dim_{\mathbb{F}}(V)$ odd) or $\mathrm{Sp}(V)$, and*

 (b) *$s = \Omega(n)$ and $\dim_{\mathbb{K}_s}(V) = 1$; and*

 (c) *$\ell(G) = 2\Omega(n) + \Omega(|T|)$ where $T = \rho(\mathbb{K}_s) \cap G$; or*

(2)

 (a) *$E(G) = \Omega^-(V)$ with $n = \dim_{\mathbb{F}}(V)$ even; and*

 (b) *$s = \Omega(n) - 1$ and $\dim_{\mathbb{K}_s}(V) = 2$; and*

 (c) *$\ell(G) \leq 2\Omega(n) - 1 + \Omega(|T|)$ where $T \cong \mathrm{SO}_2^-(\mathbb{K}_s) \cong \mathrm{GU}_1(\mathbb{K}_s)$.*

Proof: Suppose M_i and $\mathbb{K}_i$ are as described for $1 \leq i \leq s$ with $\dim_{\mathbb{K}_s}(V) = 1$ if $E(G)$ is not orthogonal and $\dim_{\mathbb{K}_s}(V) = 2$ if $E(G)$ is orthogonal. If $E(G) = \mathrm{SU}(V)$, the results from Section 4.3 of [15] together with our hypotheses easily imply $\dim_{\mathbb{F}}(V)$ odd. The maximality of M_i in M_{i-1} implies that $(\mathbb{K}_i : \mathbb{K}_{i-1}) = p_i$ is a prime divisor of $\dim_{\mathbb{K}_{i-1}}(V)$. Thus $s = \Omega(n)$ or $\Omega(n) - 1$. Moreover in the first case

$$M_s = (\mathrm{GL}_1(\mathbb{K}_s) \cap G) : \langle x_s \rangle,$$

where $\langle x_s \rangle \cong \mathrm{Gal}(\mathbb{K}_s/\mathbb{F}) \cong \mathbb{Z}_n$. Thus (b) and (c) hold in this case. If $s = \Omega(n)-1$, then

$$M_s = (O_2^{\epsilon}(\mathbb{K}_s) \cap G) : \langle x_s \rangle,$$

where again $\langle x_s \rangle \cong \mathrm{Gal}(\mathbb{K}_s/\mathbb{F}) \cong \mathbb{Z}_n$. Moreover it is clear from Section 4.3 of [15] that $\epsilon = -$, $E(G) = \Omega^-(V)$ and, as $\dim_{\mathbb{K}_s}(V)$ divides $\dim_{\mathbb{F}}(V)$, n is even. Thus (2) holds in this case.

Thus if the proposition fails, then from Table 4.3.A of [15] we conclude that either $E(M_s)$ is a quasisimple classical linear group or $E(M_s) = \Omega_4^+(\mathbb{K}_s)$ (the case $\dim_{\mathbb{K}_s}(V) = 2$ with M_s of type $\Omega_2^{\pm}(\mathbb{K}_s)$ having been treated above). In the latter case, by induction, we must have $s = 1$, $G = \Omega_{4r}^+(\mathbb{F})$ where $(\mathbb{K} : \mathbb{F}) = r$ is prime. We shall eliminate this case first.

By Lemma 2.9 and Table 3.5.E of [15], if r is odd, we may assume $G = O_{4r}^+(\mathbb{F})$. Then $M = O_4^+(q^r) \cdot \langle x \rangle$, where $\langle x \rangle \cong \mathrm{Gal}(\mathbb{K}/\mathbb{F}) \cong \mathbb{Z}_r$. Moreover, by Proposition 2.7(b), if $L_i \lhd \lhd M$ with $L_i \cong \mathrm{SL}_2(\mathbb{K})$, then L_i has hyperbolic length, whence $\ell(L_i) = \Omega(q^r + 1) + 2$, where $|\mathbb{K}| = q^r$. Thus by the structure of M, $\ell(G) = \ell(M) + 1 = 2\Omega(q^r + 1) + 7$. Now consider the chain

$$O_2^-(\mathbb{K}) \cdot \langle x_1 \rangle \times O_2^-(\mathbb{K}) \cdot \langle x_2 \rangle \subset O_{2r}^-(\mathbb{F}) \times O_{2r}^-(\mathbb{F}) \subset O_{2r}^-(\mathbb{F}) \wr \mathrm{Sym}(2) \subset G,$$

where $\langle x_i \rangle \cong \mathrm{Gal}(\mathbb{K}/\mathbb{F}) \cong \mathbb{Z}_r$. As $O_2^-(\mathbb{K}) \cong D_{q^r+1}$, we see that $\ell(G) \geq 2\Omega(q^r + 1) + 8$, a contradiction. Finally, if $r = 2$ and $G \cong \Omega_8^+(\mathbb{F})$, we see from 15.1.8 in [1] that M is conjugate via a triality automorphism to $M_1 = N_G(\{V_1, V_2\})$ where $V = V_1 \perp V_2$ and $V_1 \cong V_2$ has Witt index 1. As $\ell(M_1) = \ell(M)$, G is split hyperbolic, contrary to assumption.

Thus, if Proposition 3.4 fails, then $E(M_s)$ is a quasisimple classical linear group. As Theorem 1.1 applies to $E(M_s)$ by assumption and as G does not have parabolic length, Proposition 2.7(c) implies that $E(M_s)$ has a maximal subgroup L_s with $\ell(E(M_s)) = \ell(L_s) + 1$ and with $(E(M_s), L_s)$ satisfying Conclusion (2) of Theorem 1.1. Now let $N_{s+1} = N_{M_s}(L_s)$, so that $\ell(M_s) = \ell(N_{s+1}) + 1$ by Proposition 2.8. Then $V = V_1 \perp V_2$ as a $\mathbb{K}_s[N_{s+1}]$-module and this is an admissible $\mathbb{K}_s$-splitting of V. Now define N_i, $1 \leq i \leq s$, to be $\mathrm{Stab}_G(\{V_1, V_2\})$ where the V_i are regarded as $\mathbb{K}_{s-1}$-spaces. Then we have a proper chain

$$G \supset N_1 \supset N_2 \supset \cdots \supset N_s \supset N_{s+1}$$

with $\ell(N_{s+1}) = \ell(G) - (s+1)$ and with N_1 the stabilizer of an admissible splitting of the $\mathbb{F}$-space V, contrary to assumption. ∎

LEMMA 3.5: *E acts absolutely irreducibly on V.*

Proof: Suppose not. Then by Lemma 3.2, M is the stabilizer of a $\mathbb{K}$-linear structure on V with $\mathbb{K}$ a proper field extension of $\mathbb{F}$. Now by Proposition 3.4, $E(G)$ is one of $\mathrm{SL}(V)$, $\mathrm{SU}(V)$ ($\dim_{\mathbb{F}}(V)$ odd), $\mathrm{Sp}(V)$ or $\Omega^-(V)$ and there is a chain of stabilizers as described. In each case, it is clear that there is a chain

$$C^*\colon G \supseteq M^* \supseteq \cdots \supseteq \{e\}$$

with M^* an admissible field extension stabilizer and with $\ell(C^*) = \ell(G)$, contrary to assumption. ∎

The next two lemmas treat the case where $E(M)$ acts tensor decomposably on V. They parallel Lemmas 3.1 and 3.2.

LEMMA 3.6: *Suppose V is tensor-induced as an $\mathbb{F}[M]$-module. Then $V = V_1 \otimes V_2$ with (V_1, f_1) isometric to (V_2, f_2) (here $f = f_1 \otimes f_2$ is the form associated to G).*

Proof: Suppose not. Then $V = V_1 \otimes \cdots \otimes V_s$ as an $\mathbb{F}[M]$-module with $s \geq 3$. If $s = s_1 + s_2$ is a proper dyadic splitting with $s_1 > s_2$, set

$$W_1 = V_1 \otimes \cdots \otimes V_{s_1}, \quad W_2 = V_{s_1+1} \otimes \cdots \otimes V_s. \tag{3}$$

If $s = 2^r$, set

$$W_1 = V_1 \otimes \cdots \otimes V_{\frac{s}{2}}, \quad W_2 = V_{\frac{s}{2}+1} \otimes \cdots \otimes V_s. \tag{4}$$

As in Lemma 3.1, we are done provided $\mathrm{Stab}_G(W_i) \neq \mathrm{Stab}_M(W_i)$ for $i = 1$ or 2. As $s > 2$ and $s_1 > 1$, this is immediate if $\mathrm{Stab}_G(W_1)$ acts tensor indecomposably on W_1. As $\dim_{\mathbb{F}}(V_i) \geq 2$, this is true unless W_1 is a 4-dimensional orthogonal space of maximal Witt index and V_i is a 2-dimensional symplectic space for all

i. As $s_1 > s_2$, we are reduced to the cases $G \cong \mathrm{Sp}_8(\mathbb{F})$ and $G = \Omega_{16}^+(\mathbb{F})$ with $\dim_{\mathbb{F}}(V_i) = 2$.

If $G \cong \mathrm{Sp}_8(\mathbb{F})$, then by Proposition 4.7.4 in [15], we have

$$M \cong 2 \cdot \left((P\,\mathrm{Sp}_2(q))^3 \cdot 2^2\right) \cdot \mathrm{Sym}(3).$$

However, if $V = U_1 \perp U_2 \perp U_3 \perp U_4$ with $\dim_{\mathbb{F}}(U_i) = 2$ and $M^* = \mathrm{Stab}_G(\{U_1, \ldots, U_4\})$, then

$$M^* \cong (\mathrm{Sp}_2(q))^4 \cdot \mathrm{Sym}(4)$$

and $\ell(M^*) = \ell(M) + 3$, a contradiction. Similarly if $G = \Omega_{16}^+(\mathbb{F})$, then by Proposition 4.7.5 in [15], we have

$$M \cong 2 \cdot \left((P\,\mathrm{Sp}_2(q))^4 \cdot 2^3\right) \cdot \mathrm{Sym}(4).$$

On the other hand, if $V = U_1 \perp \cdots \perp U_4$ with $\dim_{\mathbb{F}}(U_i) = 4$ and U_i of maximal Witt index for all i, and if $M^* = \mathrm{Stab}_G(\{U_1, \ldots, U_4\})$, then M^* contains $\Omega^+(U_1) \wr \mathrm{Sym}(4)$ and as $\Omega^+(U_1) \cong 2 \cdot ((P\,\mathrm{Sp}_2(q))^2)$, we easily have $\ell(M^*) > \ell(M)$, a final contradiction. ∎

Lemma 3.7: *E acts tensor indecomposably on V.*

Proof: Suppose on the contrary that $V \cong V_1 \otimes V_2$ as an $\mathbb{F}[E]$-module. Let $E = E_1E_2$ where $E_i = \Omega(V_i, f_i)$, $i = 1, 2$. By the results from Sections 4.4 and 4.7 of [15], we may assume that E_1 is quasisimple and if both E_1 and E_2 are quasisimple we may assume that $n_1 = \dim_{\mathbb{F}}(V_1) \geq n_2 = \dim_{\mathbb{F}}(V_2)$. Let $N = N_M(E_1) = N_M(E_2)$, so that $[M\colon N] \leq 2$ with equality only if (V_1, f_1) is isometric to (V_2, f_2).

We shall first consider the case $M \neq N$ and shall exhibit a proper subgroup H of G with $\ell(H) > \ell(M)$, contrary to $\ell(G) = \ell(M) + 1$. Let $m = \dim_{\mathbb{F}}(V_1) = \dim_{\mathbb{F}}(V_2)$, so that $n = m^2$. From Table 4.7.A in [15], noting that $t = 2$, we have $m \geq 3$. If $G \cong \mathrm{SL}_{m^2}^{\epsilon}(q)$, then

$$M \subseteq (G_1 * G_2) \cdot \langle x \rangle, \quad \text{with } G_i \cong \mathrm{GL}_m^{\epsilon}(q), \quad \langle x \rangle \cong \mathrm{Sym}(2)$$

and $-I \in G_1 \cap G_2$. As $m^2 > 2m$, G contains $H \cong \mathrm{GL}_m^{\epsilon}(q) \wr \mathrm{Sym}(2)$ acting imprimitively on a $2m$-dimensional subspace of V and $\ell(H) \geq \ell(M) + 1$. Thus by Table 4.7.A from [15], we may assume that $G \cong \Omega_{m^2}^{\epsilon}(q)$ with $\epsilon \in \{+, \circ\}$.

Suppose that $E(M) = E_1 * E_2$ with $E_1 \cong E_2 \cong \Omega_m^\epsilon(q)$. Then by 4.7.6–4.7.8 from [15],

$$\ell(M) \leq \ell(E(M)) + 6$$

and $\ell(M) = \ell(E(M)) + 2$ if m is odd. Now

$$G \supset H = (L_1 \wr \mathrm{Sym}(2)) \times L_2, \qquad \text{with } L_1 \cong \Omega_m^\epsilon(q), \quad L_2 \cong \Omega_{m^2-2m}^\epsilon(q).$$

Thus $\ell(H) - \ell(E(M)) \geq 1 + \ell(L_2)$. If $m = 3$, then $L_1 \cong \Omega_3(q) \cong L_2(q)$ and $\ell(L_2) \geq 4$. If $m > 3$, then $\Omega_4^+(q) \subseteq L_2$ and so $\ell(L_2) \geq 2\ell(L_2(q)) + 1 \geq 9$. Thus in all cases, $\ell(H) > \ell(M)$, a contradiction. Thus $G = \Omega_{m^2}^+(q)$ and $E(M) = E_1 * E_2$ with $E_i \cong \mathrm{Sp}_m(q)$, $m \geq 4$, and with $\ell(M) - \ell(E(M)) \leq 2$ by 4.7.5 of [15]. Now

$$\mathrm{Sp}_m(q) \subset \mathrm{Sp}_m(q) * \mathrm{Sp}_2(q) \subset \Omega_{2m}^+(q)$$

and so $\ell(\Omega_{2m}^+(q)) \geq \ell(\mathrm{Sp}_m(q)) + 5$. As $m \geq 4$, we have $m^2 \geq 4m$ and so

$$G \supset H \cong \Omega_{2m}^+(q) \wr \mathrm{Sym}(2)$$

and

$$\ell(H) \geq 2\ell(\mathrm{Sp}_m(q)) + 11 > \ell(M),$$

a contradiction.

Thus we have $M = N$, i.e. $E_1 \lhd M$ and E_1 quasisimple. By Propositions 2.7(c) and 2.8, there exists $L_1 \subset E_1$ with $\ell(E_1) = \ell(L_1) + 1$, $\ell(M) = \ell(N_1) + 1$, where $N_1 = N_M(L_1)$, and with either

(a) $V_1 = V_{11} \perp V_{12}$ is an admissible splitting with $L_1 = \mathrm{Stab}_{E_1}(\{V_{11}, V_{12}\})$; or
(b) $\mathbb{K}_1$ is an admissible field extension of $\mathbb{F}$ and L_1 stabilizes a $\mathbb{K}_1$-linear structure on V_1.

In case (a), N_1 stabilizes the orthogonal splitting

$$V = U_1 \perp U_2 \cong (V_{11} \otimes V_2) \perp (V_{12} \otimes V_2),$$

while in case (b), N_1 stabilizes a $\mathbb{K}_1$-linear structure on V. In any case, by the results from Sections 7.1–7.3 of [15], the full stabilizer M^* in G of such a structure is maximal in G. As $N_1 < M$, we have $N_1 < M^*$ and so $\ell(M^*) \geq \ell(M)$. Replacing M by M^*, we have reduced to a previous contradiction. ∎

If M is the centralizer of a field automorphism of G we shall refer to M as a **fixed field** subgroup of G. If M is a subgroup of type $\mathcal{C}_8$ from Theorem 1 of [1] (see also Section 4.8 of [15]) M shall be called a **fixed form** subgroup of G.

LEMMA 3.8: *M is neither a fixed field nor fixed form subgroup of G. In particular, E cannot be represented over any proper subfield of $\mathbb{F}$.*

Proof: Suppose not. According to Tables 4.5.A and 4.8.A, either E is quasisimple of classical type over a (possibly proper) subfield $\mathbb{F}_0$ of $\mathbb{F}$ or G is type L_4 and $E \cong \Omega_4^+(q)$. In the latter case we have $\ell(P) > \ell(M)$ where P is a maximal parabolic subgroup of G. Thus by Propositions 2.7(c) and 2.8, there exists $M_1 \subseteq M$ with $\ell(M) = \ell(M_1) + 1$ and either

(1) $V = U_1 \perp U_2$ is an admissible splitting and $M_1 = \mathrm{Stab}_M(\{U_1, U_2\})$; or

(2) $\mathbb{K}_0$ is an admissible field extension of $\mathbb{F}_0$ with $(\mathbb{K}_0\colon \mathbb{F}_0) = r$ and M_1 is the stabilizer of a $\mathbb{K}_0$-linear structure on U.

In case (1), we may assume that $U_i = U \cap V_i$, where $V = V_1 \perp V_2$ is an admissible splitting and $M^* = \mathrm{Stab}_G(\{V_1, V_2\})$ with $M_1 = M^* \cap M$. In case (2), we may assume that $\mathbb{K}$ is a common extension of $\mathbb{K}_0$ and $\mathbb{F}$ with $(\mathbb{K}\colon \mathbb{F}) = r$ and that M^* is an admissible stabilizer of a $\mathbb{K}$-linear structure on V with $M_1 = M^* \cap M$. Again, in either case, replacing M by M^*, we have reduced to a previous contradiction. ∎

Definition 3.9: Suppose $E \subseteq \Omega(V)$ with E a quasisimple group of Lie type acting absolutely irreducibly on V over $\mathbb{F}$, where $|\mathbb{F}| = p^{\bar{c}}$. Assume $E = E(p^a)$ where $p^a = |Z(X_\alpha)|$ for X_α a long root subgroup of E. If E is of twisted type, let τ be a non-trivial graph automorphism defining E (as fixed points of a twisted Frobenius endomorphism of an algebraic group over the algebraic closure of $\mathbb{F}_p$).

(a) Set
$$\bar{c} = \begin{cases} c & \text{if } \Omega \neq \mathrm{SU}(V), \\ 2c & \text{if } \Omega = \mathrm{SU}(V). \end{cases}$$

(b) Set
$$\bar{a} = \begin{cases} a & \text{if } \Omega \text{ is untwisted or } V \cong V^\tau, \\ a \cdot |\tau| & \text{if } \Omega \text{ is twisted and } V \ncong V^\tau. \end{cases}$$

LEMMA 3.10: *$E = E(p^a)$ is a quasisimple group of Lie type acting absolutely irreducibly and absolutely tensor indecomposably on V, with $\bar{a} = \bar{c}$ ($\bar{a}$, c, $\bar{c}$ as defined above).*

Proof: By Lemmas 3.5 and 3.7, E acts absolutely irreducibly and tensor indecomposably on V. Thus by Theorem 1 of [1], E is quasisimple and $Z(M) = Z(G)$.

Now by Proposition 2.7(b), $E = E(p^a)$ is a quasisimple group of Lie type in characteristic p and, by Lemma 3.8, the representation cannot be written over any proper subfield of $\mathbb{F}$. We adopt the notation of Definition 3.9. In particular, $|\mathbb{F}| = p^{\overline{c}}$ and $\overline{a}$ is as defined above.

Suppose the lemma is false. Then by Corollary 6 of [20] we have $r = \overline{a}/\overline{c} > 1$ and $M = N_G(\mathcal{I}_m(p^{a_1}))$, where

(1) $\mathcal{I}_m(p^{a_1}) \in \{\mathrm{SL}(m,p^{a_1}), \mathrm{SU}(m,p^{a_1}), \mathrm{Sp}(m,p^{a_1}), \Omega^\epsilon(m,p^{a_1})\}$;

(2) $n = m^{r_1}$ with $r_1 \in \left\{r, \frac{r}{|\tau|}\right\}$ and $r_1 > 1$;

(3) $a_1 \mid r_1\overline{c}$, $r_1\overline{c} \mid \overline{a}$.

Set $\overline{G} = G/Z(G)$, $\overline{M} = M/Z(G)$ and $\overline{E} = (E * Z(M))/Z(M)$. Then $\ell(\overline{M}) + 1 = \ell(\overline{G})$ and by Theorem 1 of [1], $\overline{E} \lhd \overline{M} \leq \mathrm{Aut}(\overline{E})$. Let $\mathrm{InnDiag}(\overline{E})$ denote the group of inner-diagonal automorphisms of $\overline{E}$ and $\overline{Y} = \mathrm{InnDiag}(\overline{E}) \cap \overline{M}$, so $\overline{Y}$ embeds in $\mathrm{PGL}_m(p^{a_1})$. An application of Steinberg's tensor product theorem (see Proposition 5.4.6 of [15]) shows that M can induce a group of field automorphisms of order at most r_1 or $4r_1$ according to whether $E = \mathcal{I}_m(p^{a_1})$ is untwisted or twisted. In either case we have

$$\ell(\overline{M}) \leq \ell(\overline{Y}) + \Omega(r_1) + 2.$$

From the containment $\mathrm{PGL}_m(p^{r_1\cdot\overline{c}}) \cdot \mathbb{Z}_{r_1} \subset \mathrm{PGL}_{m\cdot r_1}(p^{\overline{c}})$ we conclude

$$\ell(\overline{M}) \leq \ell(\mathrm{PGL}_{m\cdot r_1}(p^{\overline{c}})) + 1. \tag{5}$$

Now $G = \mathcal{J}_{m^{r_1}}(p^c)$, where $\mathcal{J} \in \{\mathrm{SL}, \mathrm{SU}, \mathrm{Sp}, \Omega^\epsilon\}$ and

$$\mathrm{GL}_{m\cdot r_1}(p^{\overline{c}}) \subseteq \mathcal{J}_{2mr_1+1}(p^c),$$

and so Eq. (5) together with $\overline{M}$ a longest subgroup of $\overline{G}$ forces $n = m^{r_1} \leq 2mr_1$. As $r_1 > 1$, this reduces easily to the cases:

(i) $m = 2$, $n \in \{4, 8, 16\}$, or

(ii) $m = 3$, $n = 9$, or

(iii) $m = 4$, $n = 16$.

The case $m = 2$, $n = 4$ corresponds to $L_2(q^2) = \Omega_4^-(q)$. The case $m = 2$, $n = 8$ yields the embedding (see [14])

$$\mathrm{Sp}_2(q^3) \cdot \langle\eta\rangle \leq \mathrm{Sp}_8(q), \quad \text{where } \langle\eta\rangle \cong \mathrm{Gal}\left(\mathbb{F}_{q^3}/\mathbb{F}_q\right).$$

However we have $\mathrm{Sp}_2(q^3) \cdot \langle\eta\rangle < \mathrm{Sp}_6(q) < \mathrm{Sp}_8(q)$ and so $\ell(M) < \ell(G) - 1$, a contradiction. The case $m = 2$, $n = 16$ yields the embedding

$$\mathrm{PGL}_2(q^4) \cdot \langle\eta\rangle \leq \Omega_{16}^\epsilon(q), \quad \langle\eta\rangle \cong \mathrm{Gal}\left(\mathbb{F}_{q^4}/\mathbb{F}_q\right).$$

Now $\mathrm{PSL}_2(q^4) \cong \Omega_4^-(q^2) \subset \Omega_8^-(q) \subset \Omega_{10}^{\epsilon}(q)$ and so $\ell(M) < \ell(G) - 1$ again.

If $m = 3$, $n = 9$, we have (according to [14]) the embeddings $L_3(q^2) \subseteq L_9(q)$ and $L_3(q^2) \subseteq U_9(q)$. In the first case, we have $H \cong \mathrm{GL}_6(q) \subseteq G$ with $\ell(H) > \ell(M)$. In the latter case G has a maximal parabolic subgroup P with Levi factor L containing $\mathrm{SL}_3(q^2)$ and so $\ell(P) > \ell(M)$. Thus we have a contradiction in both cases.

Finally, if $m = 4$ and $n = 16$, we have $E \subseteq \mathrm{SL}_4(q^2)$ with $\ell(M) - \ell(E) \leq 3$. As

$$\mathrm{PSL}_4(q^2) \cong \Omega_6^+(q^2) \subset \Omega_{12}^+(q) \subset \Omega_{12}^+(q) \times \Omega_4^{\epsilon}(q) \subset \Omega_{16}^{\epsilon}(q),$$

we immediately have $\ell(M) < \ell(G) - 1$, unless $G \cong \mathrm{Sp}_{16}(q)$. In that case, we have $\mathrm{GL}_4(q^2) \subset \mathrm{GL}_8(q)$, the Levi complement of a maximal parabolic subgroup of G, a final contradiction. ∎

LEMMA 3.11: *E is not a classical linear group.*

Proof: Suppose not. By Lemma 3.10, $\bar{a} = \bar{c}$. Set $\tilde{a} = a$ or $2a$ according to whether E is twisted or untwisted. As E cannot be represented over any proper subfield of $\mathbb{F}$, an application of the Steinberg tensor product theorem implies that M can induce only those field automorphisms in $\mathrm{Gal}(\mathbb{F}_{p^{\tilde{a}}}/\mathbb{F}_{p^{\bar{a}}}) \leq \mathbb{Z}_2$. Arguing as in the proof of Lemma 3.10 we conclude

$$\ell(M) \leq \ell(\mathrm{PGL}_d(p^a)) + 3,$$

where d is the dimension for the natural module for $\hat{E}$ (where $\hat{E}/Z(\hat{E}) = E$). Observe that for each $\mathcal{J} \in \{\mathrm{SL}, \mathrm{SU}, \mathrm{Sp}, \Omega^{\epsilon}\}$, there is a maximal parabolic subgroup P of $\mathcal{J}_{2d+1}(q)$, the Levi factor of which contains a subgroup of index 2 in $\mathrm{GL}_d(p^a)$. Thus

$$\ell(\overline{P}) \geq \ell(\mathrm{PGL}_d(p^a)) + 3 \geq \ell(\overline{M}),$$

and so we conclude $n \leq 2d$. Moreover, if $n = 2d$, $G = \Omega_{2d}^-(q)$ and $E \not\cong \Omega_d^{\epsilon}(q)$. Using Proposition 5.4.11 of [15], we arrive at the following list of possibilities:

(i) $E = L_2(q)$ with $\dim_{\mathbb{F}}(V) = 3$ or 4,
(ii) $E = L_3^{\epsilon}(q)$ with $\dim_{\mathbb{F}}(V) = 6$,
(iii) $E = L_4^{\epsilon}(q)$ with $\dim_{\mathbb{F}}(V) = 6$,
(iv) $E = L_5^{\epsilon}(q)$ with $\dim_{\mathbb{F}}(V) = 10$,
(v) $E = \Omega_7(q)$ with $\dim_{\mathbb{F}}(V) = 8$,
(vi) $E = \Omega_{10}^{\epsilon}(q)$ with $\dim_{\mathbb{F}}(V) = 16$.

As $\Omega_4^-(q) \cong L_2(q^2)$, we may discount the case $(L_2(q), \Omega_4^-(q))$. We may likewise discount the case $L_4^\epsilon(q) \cong P\Omega_6^\epsilon(q)$. In the other L^ϵ cases, we see from Proposition 5.4.11 of [15] that $G \cong \mathrm{SL}_n^\epsilon(q)$ and we conclude that $\ell(G) > \ell(M) + 1$. Finally, when $E = \Omega_d^\epsilon(q)$, the representation is the spin representation and by Proposition 5.4.9 of [15], the possibilities for E and G are

(a) $E = \Omega_7(q)$, $G = \Omega_8^+(q)$,

(b) $E = \Omega_{10}^+(q)$, $G = \mathrm{SL}_{16}(q)$,

(c) $E = \Omega_{10}^-(q)$, $G = \mathrm{SU}_{16}(q)$.

If (a) holds, Proposition 2.2.4 of [13] shows that $E = M$ is conjugate via a triality automorphism to a reducible subgroup of G. If either (b) or (c) hold, $E = \Omega_{10}^\epsilon(q) \subset \mathrm{SL}_{16}^\epsilon(q)$ and so $\ell(G) > \ell(M) + 1$, a final contradiction. ∎

PROPOSITION 3.12:

(1) *Let* $L = G_2(q)$ *and let* $L \leq X \leq \mathrm{Aut}(L)$ *with* $\gcd(q, 3) = 1$. *There exists a longest subgroup* M_0 *of* X *with either* $M_0 \cap L$ *a maximal parabolic subgroup of* L *or* $M_0 \cap L \cong \mathrm{SL}_3^\epsilon(q) \cdot \mathbb{Z}_2$, $\epsilon = \pm 1$.

(2) *Let* $L = {}^3D_4(q)$ *and let* $L \leq X \leq \mathrm{Aut}(L)$. *There exists a longest subgroup* M_0 *of* X *with either* $M_0 \cap L$ *a maximal parabolic subgroup of* L *or one of:*

(a) $M_0 \cap L \cong (\mathrm{SL}_2(q^3) * \mathrm{SL}_2(q)) \cdot \mathbb{Z}_2$, *or*

(b) $M_0 \cap L \cong (\mathbb{Z}_{q^2+\epsilon q+1})^2 \cdot \mathrm{SL}_2(3)$, $\epsilon = \pm 1$, *or*

(c) $M_0 \cap L \cong (\mathbb{Z}_{q^4+q^2+1}) \cdot \mathbb{Z}_4$

Proof: This follows from Theorem A of [11] and the main theorem of [12]. ∎

LEMMA 3.13: *E is an exceptional group of Lie type of Lie rank at least* 4.

Proof: Suppose not. Then, as $p > 3$, we have $E = G_2(q)$ or $E = {}^3D_4(q)$. First suppose $E = {}^3D_4(q)$. By Proposition 3.12, we may assume that M_0 is of type (a), (b), or (c). In particular, $F^*(M_0)$ cannot act both absolutely irreducibly and tensor indecomposably on V. Using the main results of [1], M_0 is properly contained in a maximal subgroup M^* of G acting absolutely reducibly or tensor decomposably on V. We are thus reduced to a previous case.

Now suppose that $M = E = G_2(q)$. Then by Proposition 3.12, we may assume that $M_0 \cong \mathrm{SL}_3^\epsilon(q) \cdot \mathbb{Z}_2$. It is well known (see [10]) that if $\dim_{\mathbb{F}}(V) \leq 14$, then $\dim_{\mathbb{F}}(V) = 7$ and we have $G = \Omega_7(q)$. As $\Omega_7(q) \subset \mathcal{I}(V)$ whenever $\dim_{\mathbb{F}}(V) \geq 14$, we may certainly assume that $G = \Omega_7(q)$. Now if P is the stabilizer in G of a maximal isotropic subspace of V, then the Levi complement of P is isomorphic to a subgroup of index 2 in $\mathrm{GL}_3(q)$. Also there exists a stabilizer N of a non-isotropic line with $N' \cong \Omega_6^-(q) \cong \mathbb{Z}_2 \cdot U_4(q)$. Now clearly either $\ell(P) > \ell(M)$ or $\ell(N) > \ell(M)$, a contradiction. ∎

PROPOSITION 3.14: *Let $\hat{E} = E(\overline{\mathbb{F}})$ be a simple adjoint algebraic group of type F_4, E_6, E_7 or E_8 (where $\overline{\mathbb{F}}$ is the algebraic closure of $\mathbb{F} = \mathbb{F}_{p^a}$). Suppose σ is a Frobenius endomorphism $\hat{E}$ with $E = E(p^a) = O^{p'}(\hat{E}_\sigma)$ a simple group of Lie type. Suppose further that $E \subset G$ with G a quasisimple classical linear group which is a minimal counterexample to Theorem 1.1. Then there exists a longest subgroup M_0 of $\hat{E}_\sigma$ satisfying one of the following:*

(1) *$M_0 = N_{\hat{E}_\sigma}(\hat{D}_\sigma)$ where $\hat{D}$ is a parabolic subgroup of $\hat{E}$; or*

(2) *$M_0 = N_{\hat{E}_\sigma}(\hat{D}_\sigma)$ where $\hat{D}$ is reductive of maximal rank in $\hat{E}$.*

Furthermore, there exists no non-trivial $\mathbb{F}[E]$-module on which $E(M_0)$ acts absolutely irreducibly and tensor indecomposably.

Proof: Suppose neither (1) nor (2) hold. According to Theorem 2 of [17], a longest subgroup M_0 must be of one of the following forms:

(a) $F^*(M_0)$ is simple; or

(b) M_0 is the centralizer of a graph, field or graph-field automorphism of $O^{p'}(\hat{E}_\sigma)$ of prime order; or

(c) $F^*(M_0)$ is as in Table III of [17] and $F^*(M_0) = O^{p'}(X_\sigma)$ where $X = X^\sigma$ is as in the last column of Table II of [17]; or

(d) $\hat{E} = E_8(\overline{\mathbb{F}})$ and $F^*(M_0)$ is either $\mathrm{Alt}(6) \times \mathrm{Alt}(5)$ or $\mathrm{Alt}(6) \times \mathrm{PSL}_2(p^a)$; or

(e) $M_0 = N_{\hat{E}_\sigma}(A)$ where A is an elementary abelian group given in Theorem 1(II) of [8].

Since E has hyperbolic length, Lemma 4.2 and Proposition 4.3 of [4] imply that there is no longest subgroup M_0 of type (d) or (e).

Suppose M is a longest subgroup of E of type (c). For each choice of $F^*(M)$, one can easily show that there exists a subgroup M_0 of maximal rank in E with $\ell(M_0) \geq \ell(M)$. We present a representative case, the others being entirely similar.

Let $E = F_4(q)$, $q = p^a$ and suppose M is a longest subgroup of E with $F^*(M) = G_2(q) \times L_2(q)$. As M can induce no non-trivial field automorphisms on $F^*(M)$, we have

$$\ell(M) \leq 1 + \ell(G_2(q)) + \ell(L_2(q)). \tag{6}$$

Since E has hyperbolic length (since (1) does not hold), Theorem 3.1 of [2] implies that each simple component of $F^*(M)$ has hyperbolic length. Thus by Lemma 4.1 of [4] together with Theorems A and B of [11], a longest chain in $G_2(q)$ passes through one of the following groups:

(i) $\mathrm{SL}_3(q){:}\,2$;
(ii) $\mathrm{SU}_3(q){:}\,2$;
(iii) $(\mathrm{SL}_2(q) * \mathrm{SL}_2(q)) \cdot 2$;
(iv) $G_2(q^{1/r})$, r a prime.

Suppose (i) or (ii) holds, so Eq.(6) becomes

$$\ell(M) \leq 3 + \ell(\mathrm{SL}_3^\epsilon(q)) + \ell(L_2(q)),$$

$\epsilon = \pm 1$. According to Table 5.1 of [16], there is a subgroup M_0 of maximal rank of the form

$$M_0 = (\mathrm{SL}_3^\epsilon(q) * \mathrm{SL}_3^\epsilon(q)) \cdot \mathbb{Z}_{(3,q-\epsilon)} \cdot 2.$$

Then

$$\ell(M_0) - \ell(M) \geq \ell(\mathrm{SL}_3^\epsilon(q)) - 2 - \ell(L_2(q)) > 0,$$

as desired.

Next suppose (iii) holds, so $\ell(M) \leq 4 + 3\ell(L_2(q))$. We choose M_0 from Table 5.1 of [16] of the form

$$2^2 \cdot P\Omega_8^+(q) \cdot \mathrm{Sym}(3).$$

In $P\Omega_8^+(q)$, there is a maximal parabolic subgroup with a Levi factor containing three copies of $L_2(q)$, so trivially we have $\ell(M_0) - \ell(M) > 0$.

Suppose (iv) holds. Inside $G_2(q^{1/r})$, a longest subgroup must be of type (i)–(iv), and so we must eventually encounter a subgroup of type (i)–(iii) in some $G_2(q^{1/z})$, $z = r_1 r_2 \cdots r_t$, $r = r_1$, r_i (not necessarily distinct) primes. In particular,

$$\ell(G_2(q)) = t + \ell(X(q^{1/z})),$$

where $X(q^{1/z})$ is a group of type (i)–(iii) defined over the subfield of order $q^{1/z}$. In each case, choose M_0 as before Then we easily see that $\ell(M_0) > \ell(M)$.

Suppose M is a longest subgroup of E of type (b). If $F^*(M)$ is of the same type as E, then by induction we may assume that a longest subgroup of M_1 of M is reductive of maximal rank. Clearly $M_1 < M_0$, where M_0 is the corresponding subgroup of E and again we are done.

Thus by [19] we may assume that $E \cong E_6^\epsilon(q)$ and $F^*(M)$ is isomorphic to ${}^2E_6(\mathbb{F}_0)$ ($\epsilon = +1$), $C_4(\mathbb{F}_0)$ or $F_4(\mathbb{F}_0)$. Suppose $F^*(M) \not\cong C_4(\mathbb{F}_0)$. Again by induction we may assume that a longest subgroup M_1 of M is reductive of maximal rank in M. By inspection of Tables 5.1 and 5.2 of [16], we see that $\ell(M_1) < \ell(M^*)$ with M^* reductive of maximal rank in G. The only subtle case is:

$$E(G) \cong E_6(q^2), \quad E \cong {}^2E_6(q), \quad E(M_1) \cong \mathrm{SL}_3(q^2) \times \mathrm{SL}_3(q).$$

Then $E(M^*) \cong \mathrm{SL}_3(q^2) * \mathrm{SL}_3(q^2) * \mathrm{SL}_3(q^2)$.

Finally we have $E \cong E_6^\epsilon(q)$, $F^*(M) \cong C_4(\mathbb{F})$. Now $[M\colon F^*(M)] = 2$. By Lemma 3.13, Theorem 1.1 holds for $C_4(\mathbb{F}) \cong P\,\mathrm{Sp}_8(\mathbb{F})$. Thus we may assume that a longest subgroup M_1 of M satisfies one of:

(i) $F^*(M_1) \cong \mathrm{Sp}_2(\mathbb{F}) * \mathrm{Sp}_6(\mathbb{F})$, or

(ii) $F^*(M_1) \cong \mathrm{Sp}_4(\mathbb{F}) * \mathrm{Sp}_4(\mathbb{F})$, or

(iii) $F^*(M_1) \cong P\,\mathrm{Sp}_4(\mathbb{K})$ with $(\mathbb{K}\colon \mathbb{F}) = 2$.

In case (i), $M_1 \subseteq \tilde{M}^\epsilon$ with $F^*(\tilde{M}^\epsilon) \cong \mathrm{SL}_2(\mathbb{F}) * \widehat{L_6^\epsilon}(\mathbb{F})$, where $\widehat{L_6^\epsilon}(\mathbb{F})$ is a 2-fold cover of $\mathrm{PSL}_6^\epsilon(\mathbb{F})$. In cases (ii) and (iii), $M_1 \subseteq \tilde{M}^\epsilon$ with $E(\tilde{M}^\epsilon) \cong \mathrm{Spin}_{10}^\epsilon(\mathbb{F})$. We remark that $\mathrm{Sp}_4(\mathbb{F}) \cong \mathrm{Spin}_5(\mathbb{F})$. In all cases, $M_1 < \tilde{M}^\epsilon$ and $\tilde{M}^\epsilon$ is reductive of maximal rank.

It remains to treat case (a). By Theorem 3 of [17] together with Corollary 4 of [18], one of the following holds:

(i) $F^*(M) = O^{p'}(\hat{E}_\delta)$, where δ is a field or graph-field automorphism of $\hat{E}$; or

(ii) $F^*(M) = O^{p'}(\hat{X}_\sigma)$, where X is as in the first column of Table II of [17]; or

(iii) $F^*(M) = L_2(K)$, where K is some field of characteristic p with $p \le 113$ if $E = E_8(q)$, $p \le 67$ if $E = E_7(q)$, and $p \le 43$ if $E = E_6^\epsilon(q)$ or $F_4(q)$. Moreover, the embedding of $F^*(M)$ does not lift to an embedding of algebraic groups.

Note that the groups of type (i) have already been treated in case (b) above.

In case (ii), we easily reduce to the case in which $F^*(M) = F_4(q)$ or $C_4(q)$ in $E = E_6^\epsilon(q)$, both of which were treated in case (b) above.

It remains to treat case (iii). Suppose M is a longest subgroup of E with $F^*(M) = L_2(K)$. By Zsigmondy's Theorem (5.2.14 of [15]), there is a prime divisor r of $p^{ka}+1$ which does not divide $p^m - 1$ for any $m < 2ka$. Hence by the order formula for E, we have that $k \le 6$ for $E \in \{F_4(q), E_6(q)\}$ and $k \le 9$ for

$E = {}^2E_6(q)$. As $F^*(M)$ must have hyperbolic length, we have

$$
\begin{aligned}
\ell(M) &\leq 1 + \Omega(ka) + \ell(L_2(p^{ak})) \\
&= 2 + \Omega(ka) + \Omega(p^{ak}+1) \\
&\leq 2 + \Omega(k) + a + \Omega(p^{ak}+1) \\
&\leq 5 + a + \Omega(p^{ak}+1).
\end{aligned} \tag{7}
$$

For $E \in \{F_4(q), E_6^\epsilon(q)\}$, we have $31 \leq p \leq 43$ and we use

$$
\begin{aligned}
\Omega(p^{ak}+1) &= \sum_s \Omega_s(p^{ak}+1) \\
&\leq \sum_{s=2}^{5} \Omega_s(p^{ak}+1) + \log_7(p^{ak}+1) \\
&\leq \sum_{s=2}^{5} \Omega_s(p^{ak}+1) + 2ka.
\end{aligned}
$$

Then using Table 1 and the remark that $\Omega_s(n) \leq n/s$, we have

$$
\Omega(p^{ak}+1) \leq 2ka + \max\left\{3 + \frac{ak}{3}, 5\right\}.
$$

Thus for $E \in \{F_4(q), E_6^\epsilon(q)\}$, Eq.(7) becomes

$$
\begin{aligned}
\ell(M) &\leq \max\left\{8 + a + \frac{7}{3}ak, 10 + 2ak + a\right\} \\
&\leq \max\{8 + 22a, 10 + 19a\}.
\end{aligned}
$$

Since $p > 3$, $\Omega(p^a - 1) \geq 2$ and so

$$
\begin{aligned}
\ell_b(E) &\geq l + l\Omega(p^a - 1) + 24a \\
&\geq 12 + 24a \\
&> \max\{8 + 22a, 10 + 19a\} \\
&\geq \ell(M),
\end{aligned}
$$

where l is the rank of E, a contradiction in all cases.

Table 1. $\Omega_r(p^n+1)$ for small primes r, p

p	$r=2$	$r=3$	$r=5$
31	5 if n odd 1 if n even	0	0
37	1	$2+\Omega_3(n)$ if n is odd 0 if n is even	$1+\Omega_5(n)$ if $n \equiv 2 \bmod 4$ 0 otherwise
41	1	$1+\Omega_3(n)$ if n is odd 0 if n is even	0
43	2 if n is odd 1 if n is even	0	$2+\Omega_5(n)$ if $n \equiv 2 \bmod 4$ 0 otherwise

Finally, if $E \in \{E_7(q), E_8(q)\}$, we may use the cruder estimate that $\Omega(p^{ak}+1) \leq ak\log_2(p)$. For $E = E_7(q)$ we have $k \leq 9$ and $p \leq 67$ and this yields

$$\ell(M) \leq 5 + a(1 + k\log_2(p)) < 5 + 60a < \ell_b(E),$$

a contradiction. For $E = E_8(q)$ we have $k \leq 15$ and $p \leq 113$, whence

$$\ell(M) \leq 5 + a(1 + k\log_2(p)) < 5 + 106a < \ell_b(E),$$

again a contradiction. Thus the first part of Proposition 3.14 is established.

Now suppose that a longest chain in E is supported by $M_0 = N_{\hat{E}_\sigma}(\hat{D}_\sigma)$ where $\hat{D}$ is reductive of maximal rank in $\hat{E}$. Let V be a non-trivial $\mathbb{F}[E]$-module on which $E(M_0)$ acts absolutely irreducibly and tensor indecomposably. Lift the embedding $E(M_0) \leq E \leq \mathcal{I}(V)$ to $\hat{D} \leq \hat{E} \leq \mathcal{I}(V \otimes \overline{\mathbb{F}})$. As $\hat{D}$ is reductive of maximal rank and $p \neq 2$, the Main Theorem of [27] yields a contradiction. ∎

LEMMA 3.15: *E is not an exceptional group of Lie type.*

Proof: Suppose not. Since $Z(G) = Z(M)$, we may reduce modulo $Z(G)$ to obtain $\ell(\bar{G}) = \ell(\bar{M}) + 1$, with $\bar{E} = EZ(M)/Z(M)$ a simple exceptional group of Lie type of rank at least 4 (by Lemma 3.13). As $\bar{E}$ cannot be represented over any proper subfield of $\mathbb{F}$, an application of the Steinberg tensor product theorem shows that $\bar{M}$ can induce, at most, a field automorphism of order 2 on $\bar{E}$. Thus $\ell(\bar{M}) \leq \ell(\bar{E}) + 3$. Since $\bar{E}$ is subnormal and quasisimple, it must have hyperbolic length. Thus by Proposition 3.14 there is a longest subgroup M_0 of $\bar{E}$ of maximal rank such that M_0 does not act both absolutely irreducibly and absolutely tensor indecomposably on V. Now $\bar{E}$ is one of $F_4(q)$, $E_6^\epsilon(q)$, $E_7(q)$, or $E_8(q)$. In addition, the degree of a minimal module for $\bar{E}$ in the natural characteristic is, respectively, 26, 27, 56 or 248. One may now inspect Tables 5.1 and 5.2 of [16] and in each case note that $\bar{G}$ contains a subgroup M^* with $\ell(M^*) \geq \ell(M_0) + 4$ and which does not act both absolutely irreducibly and absolutely tensor indecomposably on V. As $\ell(M^*) \geq \ell(M)$, we have reduced to a previous contradiction. ∎

Now Lemmas 3.13 and 3.15 contradict each other, completing the proof of Theorem 1.1.

4. The proof of Theorem 4.4

The key to all finiteness results for groups of hyperbolic length is the following asymptotic lemma of Turull and Zame.

LEMMA 4.1: *Let p be a prime. Then*

$$\lim_{m\to\infty} \frac{\Omega(p^m-1)}{m} = 0.$$

Thus there exists $C = C(p)$ such that $\Omega(p^{2r}-1) \leq Cr$ for all r.

Proof: The first statement is Theorem A of [29]. It follows that there exists $R = R(p)$ with $\Omega(p^{2r}-1) \leq r$ for all $r \geq R$. Let

$$C = C(p) = \max_{1\leq r\leq R} \Omega(p^{2r}-1).$$

Then clearly $\Omega(p^{2r}-1) \leq Cr$ for all $r \leq R$. ∎

We now apply Proposition 3.4, in conjunction with Lemma 4.1 to non-split hyperbolic groups (see Definition 3.3), yielding

PROPOSITION 4.2: *Let $p \geq 31$ be a prime and $G = G(p^m)$ a non-split hyperbolic classical subgroup of $\mathrm{GL}_n(p^{\overline{m}})$ where $\overline{m}$ is as in Definition 3.9. Then there exists a maximal torus T of G satisfying each of the following:*

(1) *Either $T = S \cap G$ where S is a Singer cycle of $\mathrm{GL}_n(p^{\overline{m}})$ or $E(G) = \Omega_4^+(p^m)$ and T is a subgroup of $\mathbb{Z}_{p^m+1} \times \mathbb{Z}_{p^m+1}$;*

(2)

$$l(G) \leq \begin{cases} 2\Omega(n) + \Omega(|T|) & \text{if } E(G) \neq \Omega_4^+(p^m), \\ 5 + \Omega(|T|) & \text{if } E(G) = \Omega_4^+(p^m); \end{cases}$$

(3)

$$l(G) \leq \begin{cases} 2\Omega(n) + \Omega(p^{\overline{m}} - 1) & \textit{if } E(G) \neq \Omega_4^+(p^m), \\ 1 + 2\Omega(p^{2m} - 1) & \textit{if } E(G) = \Omega_4^+(p^m); \end{cases}$$

(4) *there exists $K = K(p)$ such that*

$$\ell(G) \leq Km\left(n - \frac{1}{4}\right).$$

Proof: Let $C(p) = C$ be as in Lemma 4.1 and let

$$K = K(p) = 2C(p) + 4.$$

The result is easily checked for all cases where $E(G)$ is not quasisimple. When $E(G)$ is quasisimple, we may invoke Proposition 3.4 to conclude that $\ell(G) \leq 2\Omega(n) + \Omega(|T|)$, where $T = G \cap S$ for S a Singer cycle of $\mathrm{GL}_n^\epsilon(p^m)$ of order $p^{mn} - \epsilon$. As $\Omega(n) < n$, we have (by Lemma 4.1)

$$\ell(G) < 2n + Cmn \leq (C+2)mn = Km\frac{n}{2} < Km\left(n - \frac{1}{4}\right)$$

and we are done. ∎

We now extend Proposition 4.2 inductively.

LEMMA 4.3: *Let p be a prime and $G = G(p^m)$ a group of hyperbolic length with natural module of dimension n over $\mathbb{F} = \mathbb{F}_{p^{\overline{m}}}$. There exists $K = K(p)$ such that*

$$\ell(G) \leq Km\left(n - \frac{1}{4}\right).$$

Proof: By Theorem 1 of [25] and Theorem 1.1 of [5], we may assume that $p \geq 31$, as there are only finitely many groups of hyperbolic length in characteristic p, $p \leq 29$.

We proceed by induction on n and let K be as in Proposition 4.2(4). By Proposition 4.2, we may assume that $E(G)$ is quasisimple and G is split hyperbolic. Thus $\ell(G) = \ell(M) + 1$, where M stabilizes an admissible orthogonal splitting $V = V_1 \perp V_2$. Set $n_j = \dim_{\mathbb{F}}(V_j)$ and let $\mathcal{I}_j$ denote the full group of isometries induced by M on V_j, $j = 1, 2$. For each $j = 1, 2$, either $\mathcal{I}_j$ is split hyperbolic or $E(\mathcal{I}_j)$ is quasisimple. In the latter case, Proposition 2.7 implies $\mathcal{I}_j$ has hyperbolic length. Now by induction, we have

$$\begin{aligned}\ell(G) = 1 + \ell(M) &\leq 1 + 1 + \ell(\mathcal{I}_1) + \ell(\mathcal{I}_2) \\ &\leq 2 + Km\left(n_1 - \frac{1}{4}\right) + Km\left(n_2 - \frac{1}{4}\right) \\ &= 2 + Km\left(n - \frac{1}{4}\right) - \frac{1}{4}Km \\ &\leq Km\left(n - \frac{1}{4}\right),\end{aligned}$$

noting that $C(p) \geq \Omega(p^2 - 1) \geq 3$ (for $p \neq 2$) and so $K = 2C + 4 > 8$. The proof is now complete. ∎

We are now ready to prove Theorem 4.4.

THEOREM 4.4: *For each prime p, there are only finitely many (possibly 0) finite Lie type groups G in characteristic p with G of hyperbolic length.*

Proof: According to Theorem A of [25], there exists $M = M(p)$ such that $G = G(q)$ is of hyperbolic length only if $q \leq p^M$. In particular, there can be at most finitely many exceptional groups in characteristic p of hyperbolic length. Now assume G is classical of dimension n over $\mathbb{F}_q$ with $q = p^m \leq p^M$. By Lemma 4.3, there exists $K = K(p)$ with

$$\ell(G) \leq KnM$$

for all such G. On the other hand,

$$\ell(G) \geq \log_p(|G|_p) \geq \frac{n}{2}\left(\frac{n}{2} - 1\right) > \frac{n^2}{6}.$$

Hence $n^2/6 < \ell(G) \leq CnM$, i.e. $n < 6CM$. So any classical group G in characteristic p of hyperbolic type has natural module of dimension $n < 6C(p)M(p)$ and has field of definition $\mathbb{F}_q$ with $q = p^m \leq p^{M(p)}$. There are only finitely many such groups and, given the previous remark about exceptional groups, this proves the theorem. ∎

5. Length formulas

In this section we prove the following bound on the length of a classical linear group. This answers a question of L. Finkelstein.

THEOREM 5.1: *Let $G = \mathcal{I}(n,q)$ be a classical matrix group with minimal field of definition $\mathbb{F}_q$ where $q = p^m$, $q > 11$. Suppose further that G is a split BN-pair of rank $r(G)$ with $B = UH = N_G(U)$, $U \in \mathrm{Syl}_p(G)$ and H a Cartan subgroup of G. Then one of the following holds:*

(i) $r(G) + \log_p(|U|) + \Omega(|H|) \le \ell(G) \le r(G) + \log_p(|U|) + \log_2(|H|)$; *or*

(ii) $E(G) \in \{\mathrm{SU}_2(p), \Omega_3(p), \mathrm{Sp}_4(p), \Omega_4^+(p), \Omega_5(p)\}$ *for p a Mersenne prime.*

Remarks: (1) By a Cartan subgroup we mean a conjugate of the (abelian) group of diagonal matrices of G.

(2) For subgroups of $\mathrm{PGL}_n(q)$, an analogous result holds (with H replaced by its image in $\mathrm{PGL}_n(q)$ and with suitable adjustments to the list given in (ii)).

(3) Since $\mathrm{SL}_2(p) \cong \mathrm{Sp}_2(p) \cong \mathrm{SU}_2(p)$, $\mathrm{SL}_2(p)$ and $\mathrm{Sp}_2(p)$ also fail to satisfy (i). For the possibilities of $E(G)$ listed in (ii), all groups between $E(G)$ and the full isometry group fail to satisfy (i). However, in the case of $\mathrm{SL}_2(q) \le G \le \mathrm{GL}_2(p)$, G fails to satisfy (i) if and only if $[G\colon \mathrm{SL}_2(p)] \le 2$.

(4) As $O_{2n}^+(q)$ is not a split BN-pair with respect to $B = N_G(U)$, our hypotheses force $G \le \mathrm{SO}_{2n}^+(q)$.

(5) The left hand side of (i) is simply the Borel length of G, i.e. the length of a chain in G which is "longest" subject to passing through B. Thus the left hand inequality is obvious. We shall refer to the right hand side of (i) as $L(G)$. Thus we must prove that $\ell(G) \le L(G)$ except in case (ii).

(6) If $p \le 29$, G has Borel length except for $q \in \{2,3,5,7,11,19,23,23^3,29\}$ (by Theorem 1.2 of [5]), so by the previous remark (i) is immediate except in these cases. For $q \le 11$, the exact length (in terms of the Borel length) may be found in [6]. In all cases,

$$\ell(G) \le L(G) + \lambda(q)\left\lceil \frac{r(G)+2}{2} \right\rceil,$$

where $\lambda(q) = 2$ for $q = 7, 11$ and $\lambda(q) = 1$ otherwise.

We now proceed with the proof of Theorem 5.1 via a series of lemmas.

LEMMA 5.2: *Suppose that G has parabolic length and every classical component of the Levi complement of every maximal parabolic of G satisfies* (i). *Then so does G.*

Proof: Let $P = QL$ be a maximal parabolic with $\ell(G) = \ell(P) + 1$ and with $Q = O_p(P)$, L a Levi complement of P. The Cartan subgroup H of G is also the

Cartan subgroup of L. Also $r(G) = r(L) + 1$. Thus by hypothesis

$$\begin{aligned}\ell(G) &= \ell(P) + 1 \\ &\le r(L) + 1 + \log_p(|L|_p) + \log_p(|Q|) + \log_2(|H|) \\ &= r(G) + \log_p(|G|_p) + \log_2(|H|) \\ &= L(G),\end{aligned}$$

as desired. ∎

PROPOSITION 5.3: *Theorem* 5.1 *holds for* $p \le 29$.

Proof: By Remark (6), we are done unless $q \in \{2, 3, 5, 7, 11, 19, 23, 23^3, 29\}$. In this case, Theorem 1.3 of [6] gives precise formulas for $\ell(G(q))$. Using these, one checks that (i) holds for $q > 11$. ∎

LEMMA 5.4: *Let p be an odd prime. Then $\Omega(p+1) > \log_2(p-1)$ if and only if p is a Mersenne prime. Thus if the simple composition factors of G are isomorphic to $\mathrm{PSL}_2(p)$, then $\ell(G) \le L(G)$ unless p is a Mersenne prime.*

Proof: If p is a Mersenne prime then $\Omega(p+1) = \log_2(p+1) > \log_2(p-1)$.

Suppose next that p is not a Mersenne prime and $\Omega(p+1) > \log_2(p-1)$. Since $p+1 \ne 2^k$, we have $\log_2(p+1) > \Omega(p+1) > \log_2(p-1)$ and so

$$\log_2(p+1) - \log_2(p-1) > \log_2(p+1) - \Omega(p+1).$$

Since $p + 1 = 2^k m$ with $m > 1$ odd,

$$\log_2(p+1) - \Omega(p+1) = \log_2(m) - \Omega(m) \ge \log_2(m) - \log_3(m).$$

The function $f(m) = \log_2(m) - \log_3(m)$ is strictly increasing and positive on $[3, \infty)$ and so

$$\log_2\left(\frac{p+1}{p-1}\right) > \log_2(p+1) - \Omega(p+1) > \log_2(3) - 1,$$

which holds only when $p < 5$. This leaves $p = 3$, a Mersenne prime.

Finally, if $G = \mathrm{PSL}_2(p)$ we may assume G is non-split hyperbolic. Therefore

$$\ell(G) = 1 + \Omega(p+1) \le 1 + \log_2(p-1) = L(G)$$

if and only if p is not a Mersenne prime. ∎

LEMMA 5.5: *Suppose G is of non-split hyperbolic type. Then $\ell(G) \leq L(G)$ unless p is a Mersenne prime and the simple composition factors of G are isomorphic to* $\mathrm{PSL}_2(p)$.

Proof: We shall make use of the following (which are easily verified):

(i) $\Omega(n) \leq n/2$.

(ii) $\log_2(n+1) \leq \log_2(n) + 1$ for all $n \geq 1$.

(iii) $\log_2(n+1) - \log_2(n) = \log_2(1 + \frac{1}{n}) < \frac{1}{2}$ for $n \geq 3$.

Suppose first that $G = \mathrm{GL}_n(q)$ or $\mathrm{GU}_n(q)$. By Proposition 4.2

$$\ell(G) \leq 2\Omega(n) + \log_2(q^n + 1) \leq 2\Omega(n) + n\log_2(q) + 1.$$

On the other hand, as $q > 3$,

$$L(G) \geq \Omega(q)\frac{n(n-1)}{2} + \frac{n-1}{2} + n\log_2(q) - \frac{n}{2}.$$

Thus if the result fails, then

$$2\Omega(n) > \Omega(q)\frac{n(n-1)}{2} - \frac{3}{2}.$$

As $\Omega(n) \leq n/2$, we have $n = 2$ and so $G = \mathrm{GL}_2(q)$. But then

$$L(G) \geq \Omega(q) + 1 + 2\log_2(q) - \frac{1}{2}$$

and

$$\ell(G) = 2\Omega(2) + 2\Omega(q^2 - 1) < 2 + \log_2(q).$$

Thus $\Omega(q) = 1$, i.e. $q = p$ and we are done by Lemma 5.4.

As G is non-split hyperbolic, it remains to consider the cases $G = \mathrm{SO}_4^+(q)$, $q \neq p$, $G = \mathrm{Sp}_{2n}(q)$, $n \geq 2$, and $G = O_{2n}^-(q)$, $n \geq 3$. If $G \neq \mathrm{SO}_4^+(q)$, then Proposition 4.2 implies

$$\ell(G) \leq 2\Omega(2n) + \Omega(q^n + 1) \leq 2\Omega(2n) + n\log_2(q) + 1.$$

On the other hand, as $q > 3$

$$L(G) \geq \Omega(q)n(n-1) + (n-1) + n\log_2(q) - \frac{n}{2}.$$

Again, if the result fails, then

$$2\Omega(2n) > \Omega(q)n(n-1) + \frac{n}{2} - 2.$$

Using $\Omega(2n) \leq n$, we immediately get $n \leq 3$ and, using $\Omega(6) = 2$, we get $n = 2$. Then $G = \mathrm{Sp}_4(q)$ and in fact, for $q > 3$,

$$L(G) \geq 4\Omega(q) + 2 + 2\log_2(q) - 1$$

and we have

$$\ell(G) = 2\Omega(4) + \Omega(q^2+1) \leq 5 + 2\log_2(q) \leq L(G),$$

as desired.

Suppose next that $G = \mathrm{SO}_4^+(q)$, $q \neq p$ with G non-split hyperbolic. Then

$$\begin{aligned}\ell(G) = 2 + 2\ell(\mathrm{PSL}_2(q)) &= 4 + 2\Omega(q+1) \\ &\leq 2\Omega(q) + 2\log_2(q+1) \\ &\leq 2\Omega(q) + 2(\log_2(q-1)+1) \\ &= L(G),\end{aligned}$$

which completes the proof. ∎

LEMMA 5.6: *Theorem* 5.1 *holds unless q is a Mersenne prime.*

Proof: Suppose not. By the above we may assume that $p \geq 31$ and that G is a minimal counterexample and G is of split hyperbolic type. Then G has a longest subgroup M which contains a subgroup E of index 1 or 2 acting on an orthogonal decomposition $V = V_1 \perp V_2$ of the natural module V for G. Suppose that G is not $\mathrm{SO}_{4n}^+(q)$ with V_1 isometric to V_2 Then we may take $E = E_1 \times E_2$ with either $E_i = \mathrm{SO}^+(V_i)$ or $E_i = \mathcal{I}(V_i) \neq O(V_i)$, where $\mathcal{I}(V_i)$ denotes the full isometry group of V_i. Thus induction applies to E_i. If $G = \mathrm{SO}_{4n}^+(q)$ with V_1 isometric to V_2, then E has a subgroup $E_1 \times E_2$ of index 2 with $E_1 = \mathrm{SO}_{2n}^\epsilon(q) = E_2$. Let

$$\delta = \begin{cases} 1 & \text{if } G = \mathrm{SO}_{4n}^+(q) \text{ and} \\ & V_1 \text{ isometric to } V_2, \\ 0 & \text{otherwise.}\end{cases}$$

Then by induction we have

$$\ell(G) \leq \ell(E_1) + \ell(E_2) + 2 + \delta \leq L(E_1) + L(E_2) + 2 + \delta.$$

The following facts are critical and are easily checked.

(1) $r(G) \geq r(E_1) + r(E_2)$.

(2) $r(G) \geq r(E_1) + r(E_2) + 1$ in the following cases:
  (a) G of type GU_{2n}, E_1 of type GU_{2m+1}, E_2 of type GU_{2r+1},
  (b) G of type O_{2n+1}, E_1 of type O_1, E_2 of type O_{2n}^-,
  (c) G of type SO_{2n}^+, E_1 of type O_{2m}^-, E_2 of type O_{2r}^-.

(3) A Cartan subgroup of E is contained in the Cartan subgroup H of G, except in precisely the three cases listed in (2).

(4) Suppose that a Cartan subgroup H_0 of E is not contained in H. Then

$$\log_2(|H_0|) - \log_2(|H|) \leq \log_2(q+1) - \log_2(q-1) \leq 1 \quad \text{for } q \geq 3.$$

(5) $\log_p(|G|_p) \geq \log_p(|E|_p) + 2$ unless the simple composition factor of G is $\mathrm{PSL}_2(p)$.

(6) $\log_p(|\,\mathrm{SO}_{4n}^+(q)|_p) > \log_p(|E|_p) + 3$, unless $n = 1$ and $q = p$.

Facts (1)–(4) yield: $r(E_1)+r(E_2)+\log_2(|H_0|) \leq r(G)+\log_2(|H|)$ where H_0 is a Cartan subgroup of E. Then using (5) and (6) and recalling that we have already handled the case where the simple composition factors of G are isomorphic to $\mathrm{PSL}_2(p)$, we conclude that

$$L(G) \geq L(E_1) + \mathrm{L}(E_2) + 2 + \delta \geq \ell(G),$$

completing the proof. ∎

We have now reduced to the case where $q = p$ is a Mersenne prime. We shall need the following sharper bound.

LEMMA 5.7: *Let $G = \mathrm{GL}_n(q)$ where $q = p$ or p^2 with p a Mersenne prime and $p \geq 31$. Then $L(G) \geq \ell(G) + \frac{1}{5}$.*

Proof: Suppose G has hyperbolic length. By Theorem 1.1, G must be non-split hyperbolic, and so by Proposition 3.4, $\ell(G) = 2\Omega(n) + \Omega(q^n - 1)$. Suppose, on the contrary, that

$$2\Omega(n) + \Omega(q^n - 1) + \tfrac{1}{5} > n - 1 + \Omega(q)\frac{n(n-1)}{2} + n\log_2(q-1).$$

Then

$$\Omega(q^n - 1) - n\log_2(q-1) > n - 1 + \Omega(q)\frac{n(n-1)}{2} - \left(2\Omega(n) + \tfrac{1}{5}\right). \tag{8}$$

But

$$n\log_2\left(\frac{q}{q-1}\right) > \log_2\left(\frac{q^n - 1}{(q-1)^n}\right) \geq \Omega(q^n - 1) - n\log_2(q-1),$$

and since $g(q) = \log_2\left(\frac{q}{q-1}\right)$ is strictly decreasing and $q \geq 31$ we have $\frac{1}{10} > \log_2\left(\frac{q}{q-1}\right)$. Therefore Eq.(8) becomes

$$\begin{aligned}\frac{5n}{100} &> n-1+\Omega(q)\frac{n(n-1)}{2} - \left(2\Omega(n)+\frac{1}{5}\right)\\ &> n-\frac{6}{5}+\frac{n(n-1)}{2} - 2\log_2(n).\end{aligned} \tag{9}$$

As $\log_2(n) \leq n-1$, we obtain

$$0 > \frac{\Omega(q)}{2}n^2 - \frac{105+50\Omega(q)}{100}n + \frac{4}{5}. \tag{10}$$

If $q = p$ then $\Omega(q) = 1$ and Eq.(10) is seen to hold only if $n < 3$. When $q = p^2$, Eq.(10) holds only for $n < 2$.

Suppose first that $n = 2$ and $q = p$. Then $\ell(\mathrm{GL}_2(p)) = 2+\Omega(p-1)+\Omega(p+1)$ and we suppose that

$$2+\Omega(p-1)+\Omega(p+1)+\tfrac{1}{5} > 2+2\log_2(p-1).$$

In particular, we must have

$$\tfrac{1}{5} + (\Omega(p+1)-\log_2(p-1)) > (\log_2(p-1)-\Omega(p-1)). \tag{11}$$

Since $h(p) = \log_2\left(\frac{p+1}{p-1}\right) = \Omega(p+1)-\log_2(p-1)$ is a strictly decreasing function ($p \geq 31$), we have $h(p) \leq h(31) < \frac{1}{10}$. Thus Eq.(11) becomes $\frac{3}{10} > \log_2(p-1) - \Omega(p-1)$. We saw previously that $\log_2(p-1)-\Omega(p-1) = \log_2\left(\frac{p-1}{2}\right)-\log_3\left(\frac{p-1}{2}\right) > \frac{9}{5}$. Therefore

$$\frac{3}{10} > \log_2(p-1)-\Omega(p-1) \geq \log_2\left(\frac{p-1}{2}\right) - \log_3\left(\frac{p-1}{2}\right) > \frac{9}{5},$$

a contradiction.

Now suppose $n = 1$. If $\ell(\mathrm{GL}_1(q)) = \Omega(q-1)+\frac{1}{5} > \log_2(q-1)$ (for either choice of q) we must have $\Omega(p-1) > \log_2(p-1) - \frac{1}{5}$ (since $\Omega(p+1) = \log_2(p+1)$). Arguing as in the previous case we obtain $\frac{1}{5} > \log_2(p-1) - \Omega(p-1) > \frac{9}{5}$ for $p \geq 31$, a contradiction.

Finally suppose that G has parabolic length. Then

$$\ell(G) = 1+\ell(P) = 1+\ell(O_p(P))+\ell(L),$$

where $L \cong \mathrm{GL}_m(q) \times \mathrm{GL}_{n-m}(q)$. Applying induction, we obtain

$$\ell(G) < n - 1 + \Omega(q)\frac{n(n-1)}{2} - \frac{2}{5} + n\log_2(q-1),$$

and so $\ell(G) + \frac{1}{5} < \ell(G) + \frac{2}{5} < n - 1 + \Omega(q)\frac{n(n-1)}{2} + n\log_2(q-1)$, as desired. ∎

Let p be a Mersenne prime and set

$$\mathcal{B} = \{\mathrm{Sp}_2(p), \mathrm{GU}_2(p), O_3(p), \mathrm{Sp}_4(p), O_4^+(p), O_5(p)\}.$$

An easy calculation yields the following:

LEMMA 5.8: *Let $G \in \mathcal{B}$. Then*

$$\ell(G) < L(G) + 2\log_2\left(\frac{p+1}{p-1}\right) \leq L(G) + \frac{1}{5}.$$

LEMMA 5.9: *Theorem 5.1 holds for $G = \mathcal{I}(n,p)$ with p a Mersenne prime.*

Proof: We proceed by induction on $|G|$. By Lemma 5.5, we may assume that G is not of split hyperbolic type. According to Proposition 5.3, we have $p \geq 31$.

Suppose that G is of parabolic type. Let P be a maximal parabolic subgroup of G with $\ell(G) = \ell(P) + 1$. Let L be the Levi complement of P and let K and J be its components. By Lemma 5.2, some component of P lies in $\mathcal{B}$. So if $K \in \mathcal{B}$, then necessarily $J \cong \mathrm{GL}_m(q)$ with $m \geq 1$ and $q = p$ or p^2. Now by Lemmas 5.7 and 5.8,

$$\ell(K) < L(K) + \tfrac{1}{5} \quad \text{and} \quad \ell(J) \leq L(J) - \tfrac{1}{5}$$

and so $\ell(P) < L(P)$, whence as in the proof of Lemma 5.2,

$$\ell(G) < L(P) + 1 = L(G).$$

Finally suppose that G is of split hyperbolic type. Let M be a longest subgroup of G of split hyperbolic type. At worst, M has two components, both in $\mathcal{B}$, and so by Lemma 5.8, $\ell(M) < L(M) + 1$. Examining the proof of Lemma 5.6, we see that we will be done if we can strengthen Fact (5) to

$$\log_p(|G|_p) \geq \log_p(|E|_p) + 3. \tag{5*}$$

Now (5*) is easily verified to hold whenever G is split hyperbolic and $G \notin \mathcal{B}$, except when $G = \mathrm{GU}_3(p)$. In this case,

$$\ell(\mathrm{GU}_3(p)) = 3 + 3\log_2(p+1)$$

and

$$L(\mathrm{GU}_3(p)) = 1 + 3 + 2\log_2(p+1) + \log_2(p-1).$$

As $\log_2(p+1) - \log_2(p-1) \leq 1$, we have $\ell(\mathrm{GU}_3(p)) \leq L(\mathrm{GU}_3(p))$, completing the proof of Lemma 5.9. ∎

Combining Lemmas 5.6 and 5.9, we have completed the proof of Theorem 5.1. We note one further upper bound for $\ell(G)$.

THEOREM 5.10: *Let $G = \mathcal{I}(n,q)$ be a classical matrix group with minimal field of definition $\mathbb{F}_q$ where $q = p^m$ with $p \geq 31$. Suppose further that $G \not\cong \mathrm{SO}_4^+(q)$ and G is a split BN-pair of rank $r(G)$ with $B = UH$, $U \in \mathrm{Syl}_p(G)$ and H a Cartan subgroup of G. Then there exists a maximal torus T of G with*

$$r(G) + \log_p(|G|_p) + \Omega(|H|) \leq \ell(G) \leq r(G) + \log_p(|G|_p) + \Omega(|T|).$$

Proof: We proceed by induction on $|G|$. If G is non-split hyperbolic, the result follows easily from Proposition 4.2. Let M be a longest subgroup of G of either parabolic or split hyperbolic type. By induction, there is a maximal torus T_0 of M with

$$\ell(M) \leq r(M) + \log_p(|M|_p) + \Omega(|T_0|).$$

Let T be a maximal torus of G containing T_0. Clearly $\Omega(|T_0|) \leq \Omega(|T|)$ and $r(M) < r(G)$; so

$$\begin{aligned}\ell(G) = \ell(M) + 1 &\leq (r(M)+1) + \log_p(|M|_p) + \Omega(|T_0|)\\ &\leq r(G) + \log_p(|G|_p) + \Omega(|T|),\end{aligned}$$

as desired. ∎

6. Concluding remarks

We conclude with a brief discussion of longest chains from a slightly different perspective. As is clear from the statement of Theorem 5.10, the length of a classical linear group G in odd characteristic $p \geq 31$ is intimately connected with the numbers $\Omega(|T|)$ as T ranges over the maximal tori of G. What is not so evident is that, in fact, the length is determined by a struggle between $\Omega(|T|)$ and $\Omega_{p,T}(|G|)$. Here we let $\mathcal{U}_p(T)$ denote the set of all T-invariant p-subgroups of G and $\Omega_{p,T}(|G|)$ denote the maximum value of $|X|$ for $X \in \mathcal{U}_p(T)$. As T

varies, the maximum value of $\Omega_{p,T}(|G|)$ is $\Omega_p(|G|)$ which is achieved uniquely when $T = H$ is the Cartan subgroup of G.

An easy corollary of Theorem 1.1 is

PROPOSITION 6.1: *Let G be a finite quasisimple classical linear group. Then there exists a longest maximal subgroup M of G which is an overgroup of some maximal torus T of G.*

By Theorem A of [25], for all sufficiently large fields of characteristic p, the length of G is

$$\Omega_p(|G|) + \mathrm{rank}(G) + \Omega(|H|)$$

and M may be chosen to be any maximal parabolic subgroup of G. Moreover, by Theorem 4.4, for fixed p and all but finitely many G, M may be chosen to be some maximal parabolic subgroup of G. On the other hand, it follows easily from Corollary 1.3 of [29] that

PROPOSITION 6.2: *Given a classical linear group scheme G and a maximal toral scheme T, there exists a prime p and a finite field $\mathbb{K}$ of characteristic p such that a longest maximal subgroup M of $G(\mathbb{K})$ must be chosen from among the overgroups of $T(\mathbb{K})$.*

The list of maximal subgroups in Theorem 1.1 is essentially minimal subject to containing overgroups of every maximal torus of $G(\mathbb{K})$. Thus by Proposition 6.2, the list in Theorem 1.1 is in some sense as short as possible.

If one wishes to write down "universal" length formulas for finite quasi-simple classical linear groups, one must proceed as follows:

Definition 6.3: Let $G = G(p^m)$ be a finite quasisimple classical linear group of parabolic length. The parabolic subgroup P of G is called a **parabolic root** of G if there is a chain $\mathcal{C}$ of G with

(1) $\mathcal{C}$: $G = P_0 > P_1 > \cdots > P_r = P > \cdots > \{e\} = P_{\ell(G)}$ and P_i is a parabolic subgroup of G for $0 \leq i \leq r$; and

(2) If $L = O^{p'}(P/O_p(P)) = L_1 * \cdots * L_s$, then each L_j has hyperbolic length.

(Note: We wish to allow the possibility that $P = B$, in which case $L = \{e\}$ and (2) is satisfied vacuously.)

It is easy to see that we have

PROPOSITION 6.4: *Let $G = G(p^m)$ be a finite quasisimple classical linear group of parabolic length. Then G has a parabolic root P and*

$$\ell(G) = (\mathrm{rank}(G) - \mathrm{rank}(P)) + \Omega_p(|O_p(P)|) + \ell\left(P/O_p(P)\right).$$

Thus the length problem is reduced to the hyperbolic case.

Definition 6.5: Let $G = G(p^m)$ be a finite quasisimple classical linear group of hyperbolic length. The "reductive" subgroup R of G is called a **hyperbolic root** of G if there is a chain $\mathcal{C}$ of G with

(1) $\mathcal{C}$: $G = R_0 > R_1 > \cdots > R_r = R > \cdots > \{e\} = R_{\ell(G)}$ and R_i is the stabilizer of an admissible orthogonal decomposition of V for $0 \leq i \leq r$; and

(2) If $L = O^{p'}(R) = L_1 * \cdots * L_s$ with $V = V_1 \perp \ldots \perp V_s$ stabilized by R, $L_i|_{V_j} = \mathrm{id}_{V_j}$ for $i \neq j$, and each L_i is either of non-split hyperbolic type or of type $(S)L_2(3)$.

Again, we have

PROPOSITION 6.6: *Let $G = G(p^m)$ be a finite quasisimple classical linear group of hyperbolic length. Then G has a hyperbolic root R.*

Now the length of each L_i is determined by the formula in Proposition 4.2 (or ad hoc for the exceptional $(S)L_2(p)$ cases) and so $\ell(R)$ is determined and we have

$$\ell(G) = v(n_1, \ldots, n_s) + \ell(R)$$

where $n_i = \dim(V_i)$ (with V_i as in (2)) and $v(n_1, \ldots, n_s)$ is the "combinatorial" function defined as follows:

Definition 6.7: Let $G(n_1, \ldots, n_s)$ be a game whose initial position is the multi-set $\{n_1, \ldots, n_s\}$ of positive integers. A move consists of adding two of the integers to produce a new multi-set of size $s-1$. The game continues until one reaches the set $\{n\} = \{n_1 + \cdots + n_s\}$. The value of a move is 1 if the two numbers added are unequal and 2 if they are equal. The value

$$v_G(n_1, \ldots, n_s)$$

of the game $G(n_1, \ldots, n_s)$ is the sum of the values of its moves and the function v is defined as

$$v(n_1, \ldots, n_s) = \max\{v_G(n_1, \ldots, n_s)\}.$$

Clearly the game $G(n_1, \dots, n_s)$ has $s-1$ moves and so trivially

$$s - 1 \leq v(n_1, \dots, n_s) \leq 2(s-1).$$

It appears likely that even deciding if $v(n_1, \dots, n_s) \geq s$ is an NP-complete problem.

Thus the "general formula" for the length of a classical linear group entails difficulties of both an arithmetical nature (factorization of "cyclotomic integers") and a combinatorial nature (the value of $v(n_1, \dots, n_s)$). In any particular case, the determination of the length of a given classical group $G = G(p^m)$ must either proceed recursively via groups of smaller rank or must entail the determination of $\Omega(|T|)$ for all maximal tori T of G and the identification of a suitable parabolic or hyperbolic root containing T. For small primes p, this task may be expedited by some elementary properties of the function $\Omega(p^n - 1)$, in particular Zsigmondy's Theorem. This is illustrated by work of the first author in [5]. However, the computational demands become infinitely unpleasant as p approaches infinity.

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